Discrete-time Control Design with Positive Semi-Definite Lyapunov Functions

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Abstract

A useful stability analysis technique from continuous-time nonlinear systems [2] is extended to the discrete-time domain. The result is illustrated on a practical example.

Key words: Control-Lyapunov functions, discrete-time systems.

1 Introduction

The use of Lyapunov stability theory in nonlinear control design has undergone a renaissance with the use of Control Lyapunov Functions (clf) [8,4,7]. The simultaneous construction of the feedback and the candidate Lyapunov function has greatly enlarged the class of systems for which closed-loop stability properties can be systematically proved. Further flexibility in this overall design method has come through the use of positive semi-definite Lyapunov functions, which originated in [2] and has been further developed in [7]. The main advantage is a reduction in complexity in the candidate Lyapunov function that often occurs in concrete examples when the positive definite requirement is weakened to positive semi-definite [2,7].

This paper extends the analysis of positive semi-definite Lyapunov functions to discrete-time nonlinear systems. This is important because Lyapunov functions tend to be even more difficult to construct for such systems, and hence the reduction in complexity that comes about by dropping the positive definite requirement can be even more advantageous. This is illustrated on a spark ignition engine equipped with fuel injection and in-cylinder air flow actuation.

Preprint submitted to Elsevier Preprint *16 June 2000*

2 Stability with Positive Semi-Definite Lyapunov Function in Discrete Time

The key concept in establishing stability of an equilibrium point with the weaker hypothesis on the Lyapunov function is the notion of conditional stability [7]. Roughly speaking, a stability property is *conditional* to a set Z if it holds for all perturbed initial condition $x_0 \in Z$.

Consider the time-invariant system

$$x(k+1) = f(x(k)) \tag{1}$$

where $x \in R^n$, and $f : R^n \to R^n$ is continuous. For $k > 0$, let $x(k; x_0) := f^{(k)}(x_0)$, where $f^{(k)}$ denotes f composed with itself k-times. An equilibrium point $x_\mathrm{e} \in R^n$ satisfies $f(x_\mathrm{e}) = x_\mathrm{e}$.

Definition 1 Conditional stability

Let $Z \subset R^n$. An equilibrium point $x_\mathrm{e} \in Z$ of the system (1) is:

- *stable conditionally to Z, if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that*

$$||x_0 - x_\mathrm{e}|| < \delta \text{ and } x_0 \in Z \Rightarrow ||x(k; x_0) - x_\mathrm{e}|| < \epsilon, \; \forall k \geq 0.$$

- *attractive conditionally to Z, if there exists an $r > 0$ such that*

$$||x_0 - x_\mathrm{e}|| < r \text{ and } x_0 \in Z \Rightarrow \lim_{\mathrm{k}\to\infty} ||x(k; x_0) - x_\mathrm{e}|| = 0 \text{ uniformly in } x_0.$$

- *asymptotically stable conditionally to Z, if it is both stable and attractive conditionally to Z.* □

A function $V : R^n \to R$ is positive semi-definite about x_e if $V(x_\mathrm{e}) = 0$ and $V(x) \geq 0$ for $x \neq x_\mathrm{e}$. The analogue of the derivative a Lyapunov function along a solution of (1) is $\Delta V = V \circ f - V$. Based upon these definitions, the following theorem is proved in discrete time.

Theorem 2 Stability with positive semi-definite V in discrete time

Consider the time-invariant system (1). Let $x = x_\mathrm{e}$ be an equilibrium of (1) and let $V(x)$ be a continuous positive semi-definite function about[1] *x_e, such that $\Delta V \leq 0$. Let Z be the largest positively invariant set contained in $\{x | V(x) = 0\}$. If $x = x_\mathrm{e}$ is asymptotically stable conditionally to Z, then $x = x_\mathrm{e}$ is stable in the sense of Lyapunov.*

[1] i.e., $V(x_\mathrm{e}) = 0$ and $V(x) \geq 0$ for $x \neq x_\mathrm{e}$.

PROOF. *The proof follows closely the one given in [7], and is by contradiction. Suppose that* $x = x_{\rm e}$ *is unstable. Then it can be shown that for every* $\epsilon > 0$ *small enough, there exist a sequence* $(x_{\rm i})_{\rm i \geq 1} \to x_{\rm e}$ *in* R^n *and a sequence* $(k_{\rm i})_{\rm i \geq 1}$ *in* N^+ *(the positive integers) such that*

$$\forall k \in [0, k_{\rm i}),\ ||x(k; x_{\rm i}) - x_{\rm e}|| < \epsilon, \tag{2}$$

$$||x(k_{\rm i}; x_{\rm i}) - x_{\rm e}|| \geq \epsilon. \tag{3}$$

Using the continuity of f *and the fact that* $f(x_{\rm e}) = x_{\rm e}$*, it can be arranged that*

$$||x(k_{\rm i}; x_{\rm i}) - x_{\rm e}|| = \epsilon. \tag{4}$$

To show this, consider the functions

$$\bar{x}(\lambda) = \lambda x_{\rm i} + (1 - \lambda) x_e, \quad \lambda \in [0, 1] \subset R. \tag{5}$$

Then, since $||x(k_{\rm i}; \bar{x}(0)) - x_e|| = 0$ *and* $||x(k_{\rm i}; \bar{x}(1)) - x_e|| \geq \epsilon$*, there exists* $\lambda_\epsilon \in (0, 1)$ *such that* $||x(k_{\rm i}; \bar{x}(\lambda_\epsilon)) - x_{\rm e}|| = \epsilon$ *by the Intermediate Value Theorem, and* $x_{\rm i}$ *can be re-defined to be* $\bar{x}(\lambda_\epsilon)$ *so that (4) holds. In case condition (2) is not satisfied for the new* $x_{\rm i}$*,* $k_{\rm i}$ *is decremented so that (2) and (3) both hold, and the above procedure is repeated until the condition is satisfied (since* $k_{\rm i}$ *is finite, the procedure must converge). From* $x_{\rm i} \to x_e$*, (4), and the continuity of* f *in (1), it follows that* $k_{\rm i} \to \infty$ *as* $i \to \infty$*.*
The new sequence $z_{\rm i} := x(k_{\rm i}; x_{\rm i})$ *belongs to a compact set, and thus there exists a subsequence* $z_{\rm n_i} := x(k_{\rm n_i}; x_{\rm n_i})$ *that converges to* $z \in R^n$ *with* $||z - x_{\rm e}|| = \epsilon$*.*
The next step is to construct pre-images of z *with certain properties. Since* $k_{\rm i} \to \infty$*, for every* $K \in N^+$*, there exists* $I_K < \infty$ *such that the sequence* $z^K_{\rm n_i} := x(k_{\rm n_i} - K; x_{\rm n_i})$*,* $i \geq I_K$ *is well-defined. By (2),* $||z^K_{\rm n_i} - x_{\rm e}|| < \epsilon$*, and thus there exists a convergent subsequence; denote the limit by* z^{-K}*. By construction,* $||z^{-K} - x_{\rm e}|| \leq \epsilon$*. By the continuity of* f*, it follows that* $f^{(K)}(z^{-K}) = z$*. It is now shown that* z *and* z^{-K} *belong to* Z*. Since* V *is continuous and non-increasing along solutions,* $V(z) := \lim_{\rm i \to \infty} V(x(k_{\rm n_i}; x_{\rm n_i})) \leq \lim_{\rm i \to \infty} V(x_{\rm n_i}) = 0$*, where the last equality used the facts that* $x_{\rm n_i} \to x_{\rm e}$ *along with* $V(x_{\rm e}) = 0$*. Since* V *is non-increasing along solutions,* $V(z) = 0$ *implies* $z \in Z$*. The same argument shows that* $z^{-K} \in Z$*.*
In summary, it has been shown that for every $\epsilon > 0$ *and* $K \in N^+$*, there exist points* $z^{-K} \in Z$ *and* $z \in Z$ *such that*

$$||z^{-K} - x_{\rm e}|| \leq \epsilon, \tag{6}$$

$$f^{(K)}(z^{-K}) = z, \tag{7}$$

$$||z - x_{\rm e}|| = \epsilon. \tag{8}$$

It remains to prove that (6) - (8) cannot hold if the equilibrium x_{e} is asymptotically stable conditionally to Z. Because $\epsilon > 0$ can be chosen arbitrary small, it can be assumed without loss of generality that for any initial condition $x_0 \in Z$ with $||x_0 - x_{\mathrm{e}}|| \leq \epsilon = ||z - x_{\mathrm{e}}||$, the solution of (1) converges to x_e. So, there exists a constant $K = K(\epsilon) > 0$, independent of x_0, such that $||x(K; x_0) - x_{\mathrm{e}}|| \leq \frac{\epsilon}{2}$. Because of (6) - (8), one possible choice for x_0 is z^{-K}. But then $\frac{\epsilon}{2} \geq ||x(K; z^{-K}) - x_{\mathrm{e}}|| = ||x(K-K; z) - x_{\mathrm{e}}|| = ||z - x_{\mathrm{e}}|| = \epsilon$ which is a contradiction. □

3 Example

Engine control problems are a rich source of discrete-time systems because the models are typically sampled synchronously with combustion events [1,9]. This section presents an application of the theorem to a simplified model of an engine with fuel injection and an electro-hydraulically controlled cam [6]. The engine is representative of proto-type spark ignition engine designs. The electro-hydraulically controlled cam allows the direct control of the mass air flow rate into the cylinders, with the result that cylinder mass air flow rate can be regarded as an independent control input. The control objective is to asymptotically regulate engine brake torque to a defined value, $T_{\mathrm{b\_ref}}$, while regulating air-to-fuel ratio (A/F) to stoichiometry [2], A/F_{stoic}. Near stoichiometry, the brake torque (Nm) is approximated by

$$T_{\mathrm{b}} = 410\ CAC - 3(A/F - A/F_{\mathrm{stoic}}) - 37.5 + 0.04N - 0.0001N^2 \quad (9)$$

where CAC and N represent cylinder air charge in grams, and engine speed in rad/sec respectively. The cylinder air charge can be approximated by cylinder mass air flow rate $\times$ T, where T is elapsed time for the intake stroke, which is equal to π/N seconds.

A simplified model of the system, with the outputs already augmented with integrators for asymptotic tracking of set-points, is depicted in Figure 1. The

[2] The three-way catalytic converter used in modern vehicles to reduce emissions of CO,HC and NOx is only effective if the air-to-fuel ratio is maintained at stoichiometry.

model equations are

$$
\begin{aligned}
x_1(k+1) &= u_2(k) \\
q_1(k+1) &= q_1(k) + T(T_{\mathrm{b}} - T_{\mathrm{b\_ref}}) \\
&= q_1(k) + T(410Tu_1(k) - 3(x_1(k)u_1(k) - A/F_{\mathrm{stoic}}) + \delta(N) - T_{\mathrm{b\_ref}}) \\
q_2(k+1) &= q_2(k) + T(A/F - A/F_{\mathrm{stoic}}) \\
&= q_2(k) + T(x_1(k)u_1(k) - A/F_{\mathrm{stoic}})
\end{aligned}
$$

where (10)

u_1 : Cylinder mass air flow rate

u_2 : Inverse (amount of) mass fuel flow rate

$T_{\mathrm{b\_ref}}$: Reference brake torque (Nm)

T : Intake event duration, $\frac{\pi}{N}$ (sec)

$\delta(N)$: Portion of brake torque that is independent of A/F and cylinder air charge ($= -37.5 + 0.04N - 0.0001N^2$)

A controller will now be designed on the basis of a positive semi-definite Lyapunov function. Due to the common terms in $q_1(k+1)$ and $q_2(k+1)$, it is natural to choose a candidate Lyapunov function as

$$V_{\mathrm{L1}}(k) = V_1^{\,2}(k) = (q_1(k) + 3q_2(k))^2 \tag{11}$$

so that as long as one of q_1 or q_2 can be shown to be bounded, the other one will be bounded also. The difference equation of this Lyapunov function is given by

$$
\begin{aligned}
V_{\mathrm{L1}}(k+1) - V_{\mathrm{L1}}(k) &= (V_1(k+1) - V_1(k))(V_1(k+1) + V_1(k)) \\
&= T(410Tu_1(k) + \delta(N) - T_{\mathrm{b\_ref}})(V_1(k+1) + V_1(k)).
\end{aligned} \tag{12}
$$

Choosing the control law as

$$u_1(k) = \frac{1}{410T}\left(T_{\mathrm{b\_ref}} - \delta(N) - c_1\frac{1}{T}V_1(k)\right) \tag{13}$$

with appropriate gain c_1 results in

$$V_1(k+1) - V_1(k) = -c_1V_1(k) \tag{14}$$

and makes the difference equation of Lyapunov function V_{L1} negative semi-definite:

$$V_{\mathrm{L1}}(k+1) - V_{\mathrm{L1}}(k) = -c_1(2-c_1){V_1}^2(k). \tag{15}$$

In the next step, another candidate Lyapunov function with parameter κ is chosen to force one of the integral states, q_2, to be bounded relative to the state x_1:

$$V_{\mathrm{L2}}(k) = {V_2}^2(k) = (\kappa q_2(k) + x_1(k))^2. \tag{16}$$

Thus, if it can later be proven that any one of x_1, q_1 or q_2 is bounded, then all of them are. The difference equation of Lyapunov function V_{L2} is given by

$$\begin{aligned} V_{\mathrm{L2}}(k+1) - V_{\mathrm{L2}}(k) &= (V_2(k+1) + V_2(k))(V_2(k+1) - V_2(k)) \\ &= (V_2(k+1) + V_2(k))(\kappa T(x_1(k)u_1(k) - A/F_{\mathrm{stoic}}) + u_2(k) - x_1(k)). \end{aligned} \tag{17}$$

Choosing the control law with appropriate gain c_2

$$u_2(k) = -\kappa T(x_1(k)u_1(k) - A/F_{\mathrm{stoic}}) + x_1(k) - c_2 V_2(k) \tag{18}$$

results in

$$V_2(k+1) - V_2(k) = -c_2 V_2(k) \tag{19}$$

and makes the difference equation of Lyapunov function V_{L2} negative semi-definite:

$$V_{\mathrm{L2}}(k+1) - V_{\mathrm{L2}}(k) = -c_2(2-c_2){V_2}^2(k). \tag{20}$$

A composite, positive semi-definite Lyapunov function for the model (10) is given by

$$V_{\mathrm{L}}(k) = V_{\mathrm{L1}}(k) + V_{\mathrm{L2}}(k) = {V_1}^2(k) + {V_2}^2(k). \tag{21}$$

Then the difference equation of Lyapunov function V_{L} with inputs (13) and (18) becomes negative semi-definite

$$V_{\mathrm{L}}(k+1) - V_{\mathrm{L}}(k) = -c_1(2-c_1){V_1}^2(k) - c_2(2-c_2){V_2}^2(k). \tag{22}$$

The goal now is to understand what (22) implies about the stability of the model (10). When $V_{\text{L}1}$ and $V_{\text{L}2}$ are both equal to zero, the control signals become

$$
\begin{aligned}
u_1(k) &\longrightarrow \tfrac{T_{\text{b\_ref}} - \delta(N)}{410T} \\
u_2(k) &\longrightarrow \left(1 - \kappa\tfrac{T_{\text{b\_ref}} - \delta(N)}{410}\right) x_1(k) + \kappa T \cdot A/F_{\text{stoic}}
\end{aligned}
\tag{23}
$$

The parameter κ is now chosen so that

$$
x_1(k+1) = \left(1 - \kappa\frac{T_{\text{b\_ref}} - \delta(N)}{410}\right) x_1(k) + \kappa T \cdot A/F_{\text{stoic}} \tag{24}
$$

is asymptotically stable. This can be achieved with

$$
|1 - \kappa\frac{T_{\text{b\_ref}} - \delta(N)}{410}| < 1. \tag{25}
$$

Under this condition, the states of (10) are asymptotically stable conditionally to the largest positively invariant set contained in $Z = \{x|V_{\text{L}}(x) = 0\}$. By the theorem of the previous section, the states of (10) are bounded, and thus by LaSalle's Theorem [5], they approach the largest positively invariant set contained in $W = \{x|\Delta V_{\text{L}}(x) = 0\}$. From (21) and (22), $W = Z$. In turn, it follows that the control signals converge to (23), and consequently, the states converge to constant values. This then gives that the steady state torque and A/F errors are zero.

Since the states are converging to a constant value, this must be an equilibrium point. At no point in the analysis was the explicit value of the equilibrium point required. This is a major benefit of working with a positive semi-definite Lyapunov function: a positive definite Lyapunov function would have required the explicit value of the equilibrium point. This point is made even more clearly in [3,10], where the above analysis is extended to a full (ten) state model of the system.

A simulation of the controller (13) and (18) in closed-loop with the model (10) is shown in Figure 2. The engine speed was set to 150 rad/sec, the parameter κ was chosen to be 1, and the constant c_1 and c_2 were each set equal to be 0.5. The stoichiometric value of air-to-fuel ratio was assumed to be 14.6. It is seen that the states of the system converge, and that the controller achieves zero steady state error.

Acknowledgments

This work was supported by an NSF GOALI grant, ECS-9631237, with matching funds from Ford Motor Company.

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FIGURES

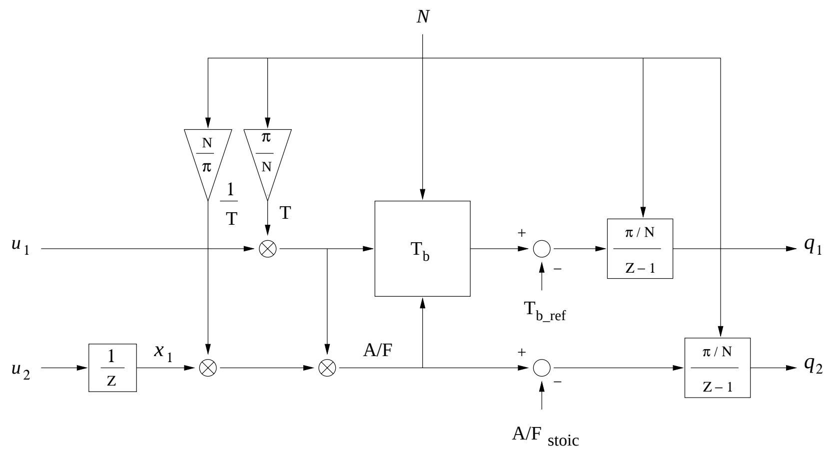

Fig. 1. Simplified engine model.

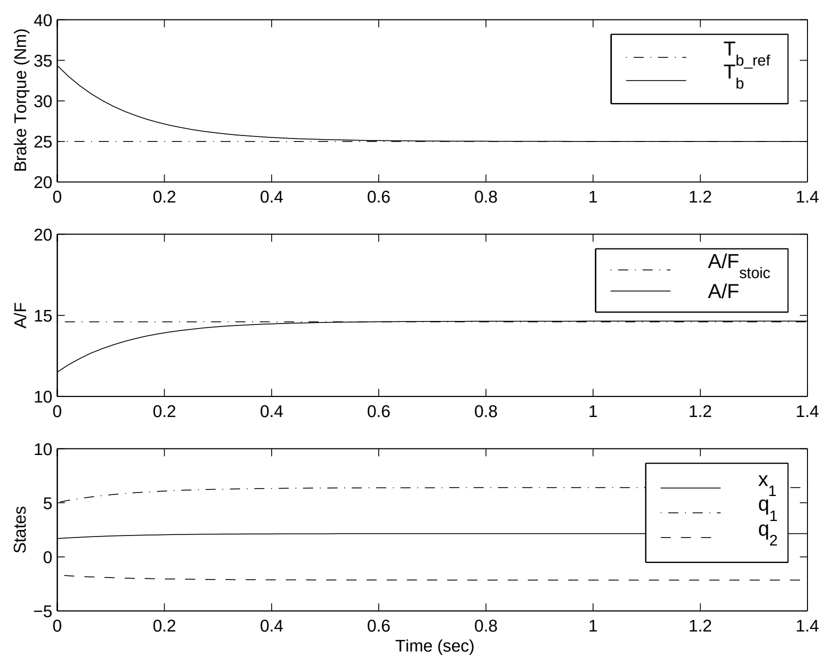

Fig. 2. Simulation result at constant engine speed 150 rad/sec. A/F_{stoic} is equal to 14.6.