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The Extended Kalman Filter as a Local Asymptotic Observer for Discrete-Time Nonlinear Systems*

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Abstract

The convergence aspects of the extended Kalman filter, when used as a deterministic observer for a nonlinear discrete-time system, are analyzed. Systems with nonlinear output maps are treated, and the conditions needed to ensure the uniform boundedness of the error covariances are related to the observability properties of the underlying nonlinear system. Furthermore, the uniform asymptotic convergence of the observation error is established whenever the nonlinear system satisfies an observability rank condition and the states stay within a convex compact domain. This last result provides a theoretical foundation for this classic, approximate nonlinear filter.

Key words: discrete-time nonlinear systems, observability rank condition, extended Kalman filter, uniform asymptotic stability

AMS Subject Classifications: 93C55, 93D20, 93E11

1 Introduction

Designing an observer for a nonlinear system is quite a challenge. Thus, as a first step, it is interesting to see how classical linearization techniques work with nonlinear systems and what their limitations are. In [4], Baras et al. describe a method for constructing observers for dynamic systems as asymptotic limits of filters. They discuss the method as applied to the linear case, and a class of nonlinear systems with linear observations[1], in the continuous-time domain. Essentially the extended Kalman filter(EKF)

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[1]See also [13] for the case of nonlinear outputs.

2 The Kalman Filter: A Global Asymptotic Observer for Linear Time-Varying Systems

It is well known that, under stochastic controllability and observability assumptions, the Kalman filter for a linear time-varying system with artificial noises can be used as a global asymptotic observer for the underlying deterministic system [9]. This fact can be also seen from the duality of a linear optimal regulator problem [18, p. 535]. In this Section, we give a new, simple proof, which is essential for setting up the analysis on nonlinear systems done in Section 3 through Section 5.

Consider a linear system:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k, \quad x_0 \text{ unknown}, \\ y_k &= C_k x_k, \end{aligned} \tag{2.1}$$

where A_k is assumed invertible[2], and consider also the associated "*noisy*" system:

$$\begin{aligned} z_{k+1} &= A_k z_k + B_k u_k + N w_k, \\ \xi_k &= C_k z_k + R v_k, \end{aligned} \tag{2.2}$$

where the design parameters N and R are to be chosen as positive definite matrices. Then the Kalman filter equations for (2.2) are given as follows [1].

Measurement update:

$$\begin{aligned} \hat{x}_k &= \bar{x}_k + K_k(\xi_k - C_k \bar{x}_k), \\ P_k^{-1} &= \bar{P}_k^{-1} + C_k^T (RR^T)^{-1} C_k, \end{aligned} \tag{2.3}$$

Time update:

$$\begin{aligned} \bar{x}_{k+1} &= A_k \hat{x}_k + B_k u_k, \\ \bar{P}_{k+1} &= A_k P_k A_k^T + NN^T, \end{aligned} \tag{2.4}$$

$$K_k = P_k C_k^T (RR^T)^{-1} = \bar{P}_k C_k^T (C_k \bar{P}_k C_k^T + RR^T)^{-1}$$

where $\bar{P}_k$ and P_k are the *a priori* and *a posteriori* error covariances, and $\bar{x}_k$ and $\hat{x}_k$ the *a priori* and *a posteriori* estimates of the state at time k, respectively. The filter is initiated with $\bar{x}_0$ and $\bar{P}_0$; $\bar{P}_0$ is used as a design parameter, assumed also positive definite.

[2] This assumption can be relaxed to singular state transition matrices if a linear system is considered [25]. Toward nonlinear systems, however, we make this stronger assumption here.

Remark 2.2 The conditions (2.11) and (2.12) imply that the "noisy" system (2.10) is stochastically controllable and observable [3]. It is easily seen that the positive definiteness of N implies the stochastic controllability through the condition (2.11). For linear systems, this requirement can be weakened to stabilizability [2] or even to nonstabilizability under a few more assumptions [7]. For the ease of presentation and the nonlinear systems to be considered later, however, we use this stronger assumption here. On the other hand, let's take $R = I$, R being a design parameter; then condition (2.12) is satisfied if the deterministic part of the system (2.10), i.e., the pair (A_k, C_k), is uniformly completely observable [18]. □

Remark 2.3 Under the above conditions, it can be also shown that $\bar{P}_k$ is bounded from above and below. Indeed, from (2.4),

$$||\bar{P}_k|| \le (\alpha_1 + 1/\beta_1)||A||^2 + ||N||^2.$$

Also, from (2.3),

$$\bar{P}_k \ge P_k \ge \frac{1}{\beta_2 + 1/\alpha_2} I.$$

Therefore,

$$\frac{1}{\beta_2 + 1/\alpha_2} I \le \bar{P}_k \le \{(\alpha_1 + 1/\beta_1)||A||^2 + ||N||^2\} I.$$

It is obvious that P_k^{-1} and $\bar{P}_k^{-1}$ are both bounded from above and below. □

Remark 2.4 Deyst and Price [9] have also shown that, under stochastic controllability and observability assumptions, the homogeneous filter equations of the *a posteriori* estimates are uniformly asymptotically stable. Since, in [9], $A_k^T A_k$ is assumed bounded from above and below in norm, and $\bar{x}_{k+1} = A_k \hat{x}_k$ when the control variable is not considered, uniform asymptotic stability also holds for the homogeneous filter equations of the *a priori* estimates (2.5), which is exactly the same as the error dynamics (2.7).

Baras et al. [4] have also obtained bounds for the error covariances in continuous-time via dual optimal control problems under some "stronger" observability and controllability assumptions (see conditions (28) and (29) in [4]) and used the bounds to show the convergence of the error. Similar methods yield bounds for the error covariances in discrete-time. Bounds for the case of linear time invariant systems are explicitly shown in [26], but in this case they follow simply from the detectability of the pair (A, C) and the invertibility of A. In Section 4 we will discuss how the observability of where

$$\begin{aligned} K_k &:= \bar{P}_k C_k^T (C_k \bar{P}_k C_k^T + RR^T)^{-1}, \\ A_k &:= \frac{\partial f}{\partial x}(\hat{x}_k), \\ C_k &:= \frac{\partial h}{\partial x}(\bar{x}_k). \end{aligned}$$

The Riccati equations for the error covariances are the same as in (2.8) and (2.9) with the above matrices.

To begin with, we make the following assumptions for setting up the analysis. Section 4 addresses how Assumption 3.1.1 is implied by an observability property of (3.1); the other conditions are addressed in Section 5.

Assumption 3.1

1. *The linearized system along the estimated trajectory of the extended Kalman filter is uniformly observable, that is, (A_k, C_k) of (3.3) and (3.4) satisfies the uniform observability condition.*

2. *$A(x) := \frac{\partial f}{\partial x}(x)$ is invertible at each $x \in \mathbf{R}^n$.*

3. *The following norms are bounded;*

$$||A|| := \sup_{x \in \mathbf{R}^n} ||A(x)||, \quad ||A^{-1}|| := \sup_{x \in \mathbf{R}^n} ||[A(x)]^{-1}||,$$

$$||H|| := \sup_{x \in \mathbf{R}^n} ||R^{-1}\frac{\partial h}{\partial x}(x)||, \quad |||D^2 f||| = \sup_{x \in \mathbf{R}^n} |||D^2 f(x)|||,$$

$$|||D^2 h||| = \sup_{x \in \mathbf{R}^n} |||D^2 h(x)|||.$$

4. *Let $g(x,y) := h(x) - h(y) - \frac{\partial h}{\partial x}(x)(x-y)$, and suppose that there exists $g < \infty$ such that $|g(x,y)| \le g|||D^2 h|||\ |x-y|^2$ for all $x, y \in \mathbf{R}^n$.*

Assumption 3.1.1 implies that the error covariances are uniformly bounded. Thus let $0 < q, p_1 < \infty$ be the corresponding bounds for error covariances, that is, $||\bar{P}_k|| \le q$ and $||P_k^{-1}|| \le p_1$ for all $k \ge 0$. For later use we derive a few more bounds. From (3.3)

$$\bar{P}_M^{-1} = P_M^{-1} - H_M^T H_M,$$

thus giving

$$||\bar{P}_M^{-1}|| \le ||P_M^{-1}|| + ||H||^2 \le p_1 + ||H||^2 := p.$$

where

$$\begin{aligned} B_k &= \int_0^1\int_0^1 D^2 f(\hat{x}_k + rs\tilde{e}_k)s\tilde{e}_k drds \\ l_k &= -A_k K_k g_k + B_k \tilde{e}_k. \end{aligned}$$

Hence,

$$\begin{aligned} e_{k+1}^T \bar{P}_{k+1}^{-1} e_{k+1} &= (e_k^T (I - K_k C_k)^T A_k^T + l_k^T)\bar{P}_{k+1}^{-1}(A_k(I - K_k C_k)e_k + l_k) \\ &= e_k^T (I - K_k C_k)^T A_k^T \bar{P}_{k+1}^{-1} A_k (I - K_k C_k) e_k \\ &\quad + l_k^T \bar{P}_{k+1}^{-1} A_k \times (I - K_k C_k) e_k \\ &\quad + e_k^T (I - K_k C_k)^T A_k^T \bar{P}_{k+1}^{-1} l_k + l_k^T \bar{P}_{k+1}^{-1} l_k. \end{aligned}$$

Using the linear results,

$$\begin{aligned} \Delta V(k, e_k) &= e_{k+1}^T \bar{P}_{k+1}^{-1} e_{k+1} - e_k^T \bar{P}_k^{-1} e_k \\ &\leq -e_k^T \bar{P}_k^{-1}(P_k^{-1} + A^T(NN^T)^{-1}A)^{-1}\bar{P}_k^{-1} e_k + l_k^T \bar{P}_{k+1}^{-1} A_k (I \\ &\quad - K_k C_k) e_k + e_k^T (I - K_k C_k)^T A_k^T \bar{P}_{k+1}^{-1} l_k + l_k^T \bar{P}_{k+1}^{-1} l_k. \end{aligned}$$

With the definition of $g_k = g(x_k, \bar{x}_k)$, since

$$\begin{aligned} |\tilde{e}_k| &= |(I - K_k C_k) e_k - K_k g_k| \\ &\leq \|I - K_k C_k\|\, |e_k| + \|K_k\|\, |g_k| \\ &\leq (pq + \delta g |||D^2 h|||\, |e_k|)|e_k|, \end{aligned}$$

and

$$\begin{aligned} \|B_k\| &= \|\int_0^1\int_0^1 D^2 f(\hat{x}_k + rs\tilde{e}_k)s\tilde{e}_k drds\| \\ &\leq \int_0^1\int_0^1 |||D^2 f|||sdrds|\tilde{e}_k| = \frac{1}{2}|||D^2 f|||\, |\tilde{e}_k|, \end{aligned}$$

it follows that

$$\begin{aligned} |l_k| &= |-A_k K_k g_k + B_k \tilde{e}_k| \\ &\leq \phi(|e_k|, |||D^2 f|||, |||D^2 h|||)|e_k|^2 \end{aligned}$$

and

$$\begin{aligned} & l_k^T \bar{P}_{k+1}^{-1} A_k (I - K_k C_k) e_k + e_k^T (I - K_k C_k)^T A_k^T \bar{P}_{k+1}^{-1} l_k + l_k^T \bar{P}_{k+1}^{-1} l_k \\ &\quad \leq \|\bar{P}_{l+1}^{-1}\|\, |l_k|(2\|A\|\, \|I - K_k C_k\|\, |e_k| + |l_k|) \\ &\quad \leq p|e_k|^3 \phi(|e_k|, |||D^2 f|||, |||D^2 h|||)\{2pq\|A\| \\ &\qquad + \phi(|e_k|, |||D^2 f|||, |||D^2 h|||)|e_k|\}. \end{aligned}$$

If we assume further that $A_k^T A_k \geq \nu I > 0 \ \forall k$, then condition (4.1) is equivalent to the following condition: for some η_1, η_2, $0 < \eta_1 \leq \eta_2 < \infty$,

$$\eta_1 I \leq O^T(k-M,k)O(k-M,k) \leq \eta_2 I, \tag{4.2}$$

where

$$O(k-M,k) := \begin{bmatrix} C_{k-M} \\ C_{k-M+1}A_{k-M} \\ \vdots \\ C_k A_{k-1}\cdots A_{k-M} \end{bmatrix}.$$

In order to apply this linear observability condition to the EKF (3.3) and (3.4) and, ultimately, to relate this to observability properties of the underlying nonlinear system, let's represent $O(k-M,k)$ in terms of the EKF variables in (3.3) and (3.4), i.e.,

$$O_e(\bar{x}_{k-M},\cdots,\bar{x}_k,\hat{x}_{k-M},\cdots,\hat{x}_{k-1}) := \begin{bmatrix} \frac{\partial h}{\partial x}(\bar{x}_{k-M}) \\ \frac{\partial h}{\partial x}(\bar{x}_{k-M+1})\frac{\partial f}{\partial x}(\hat{x}_{k-M}) \\ \vdots \\ \frac{\partial h}{\partial x}(\bar{x}_k)\frac{\partial f}{\partial x}(\hat{x}_{k-1})\cdots\frac{\partial f}{\partial x}(\hat{x}_{k-M}) \end{bmatrix} \tag{4.3}$$

Define the map $H: \boldsymbol{R}^n \to (\boldsymbol{R}^p)^n$ by

$$H(x) := (h(x), h(f(x)), \cdots, h(f^{n-1}(x)))^T. \tag{4.4}$$

A system is said to satisfy the ***observability rank condition*** at x_0 [24] if the rank[4] of the map H at x_0 equals n. A system satisfies the ***observability rank condition on*** $\mathcal{O}$ if this is true for every $x \in \mathcal{O}$; if $\mathcal{O} = \boldsymbol{R}^n$, then one says that the system satisfies the observability rank condition. By the chain rule,

$$\begin{aligned} \frac{\partial H}{\partial x}(x_0) &= \begin{bmatrix} \frac{\partial h}{\partial x}(x_0) \\ \frac{\partial h}{\partial x}(x_1)\frac{\partial f}{\partial x}(x_0) \\ \vdots \\ \frac{\partial h}{\partial x}(x_{n-1})\frac{\partial f}{\partial x}(x_{n-2})\cdots\frac{\partial f}{\partial x}(x_0) \end{bmatrix} \\ &=: \frac{\partial H}{\partial x}(x_0, x_1, \cdots, x_{n-1}) \end{aligned} \tag{4.5}$$

where $x_{k+1} = f(x_k)$, $k = 0, 1, \cdots, n-2$. It follows that $O_e = \frac{\partial H}{\partial x}$ if $\bar{x}_k$ and $\hat{x}_k$ are equal to the true state x_k, for $k = 0, 1, \cdots, n-1$. By continuity, we can argue that if the system (3.1) satisfies the observability rank condition, then its associated EKF satisfies the observability condition (4.2), for $M =$

[4] Recall that the rank of H at x_0 equals the rank of $\frac{\partial H}{\partial x}(x)$ evaluated at x_0.

Remark 4.3 Suppose that the system (3.1) satisfies the observability rank condition and that the output y is scalar valued. Then $\tilde{x} = H(x)$ is a local diffeomorphism. In the $\tilde{x}$- coordinates, the system (3.1) is transformed into a local, observer canonical form:

$$\begin{aligned} \tilde{x}_1(k+1) &= \tilde{x}_2(k) \\ &\vdots \\ \tilde{x}_{n-1}(k+1) &= \tilde{x}_n(k) \\ \tilde{x}_n(k+1) &= \phi(\tilde{x}_1(k), \cdots, \tilde{x}_n(k)) \\ y &= \tilde{x}_1(k). \end{aligned} \tag{4.7}$$

A simple computation shows that the linearized observability condition (4.2) is always satisfied for a system in the form (4.7); indeed, $O(k-M, k) \equiv I_n$ for $M = n-1$. This is in marked contrast to the situation analyzed in Proposition 4.1, and underlines the coordinate dependence of the extended Kalman filter in general, and the linearized observability condition (4.2) in particular.

5 Applicability of EKF as an Observer for Nonlinear Systems

In this section we seek to remove Assumption 3.1 by applying the EKF on a convex compact subset of the state space. Before we begin, a few notations are mentioned. Let $\mathcal{O}$ be a (not necessarily small) convex compact subset of $\mathbb{R}^n$, $\sim \mathcal{O}$ the complement of $\mathcal{O}$, and $\epsilon > 0$ a positive constant. Define $d(x, \sim \mathcal{O}) = inf\{|x-y| : y \in \sim \mathcal{O}\}$, and $\mathcal{O}_\epsilon = \{x \in \mathcal{O} : d(x, \sim \mathcal{O}) \geq \epsilon\}$. Since $\mathcal{O}$ is compact, $||A|| := \sup_{x \in \mathcal{O}} ||\frac{\partial f}{\partial x}(x)||$ and $||Dh|| := \sup_{x \in \mathcal{O}} ||\frac{\partial h}{\partial x}(x)||$ are bounded. Let $a = max(1, ||A||)$ and

$$\begin{aligned} b_k &= (1 + ||\bar{P}_0|| ||Dh||^2 ||R^{-1}||^2) a^k \prod_{l=1}^{k} \{1 + ||Dh||^2 ||R^{-1}||^2 \\ &\quad \times [||A||^{2l} ||\bar{P}_0|| + ||N||^2 (||A||^{2(l-1)} + ||A||^{2(l-2)} + \cdots + 1)]\}. \end{aligned}$$

First we consider a sufficient condition to keep the estimates $\bar{x}_k$ and $\hat{x}_k$ near the true state x_k over a finite time period.

Lemma 5.1 *Consider the system (3.1) and its associated EKF (3.3) and (3.4). Suppose that the following conditions hold.*

1. $x_k \in \mathcal{O}_\epsilon$, *for some* $\epsilon > 0$, $0 \leq k \leq M$.
2. $|e_0| = |\bar{x}_0 - x_0| \leq \frac{\delta}{b_M}$ *for some* $0 < \delta \leq \epsilon/2$.

Therefore, for $2 \le k \le M$,

$$|\bar{x}_k - x_k| \le b_{k-1}|e_0| \le \delta.$$

Also,

$$\begin{aligned} |\hat{x}_k - x_k| &\le (1 + \|K_k\| \cdot \|Dh\|)|\bar{x}_k - x_k| \\ &\le b_k|e_0| \le \delta. \end{aligned}$$

This completes the proof. □

Since we have conditions which keep the EKF estimates close to the true state, we can now use the results of Theorem 3.2, Proposition 4.1, and Lemma 5.1 to show the convergence of the EKF on a convex compact set without Assumption 3.1. It is only required that the system (3.1) satisfy the observability rank condition on a convex compact set $\mathcal{O}$, and that $[\frac{\partial f}{\partial x}(x)]^{-1}$ exist at each $x \in \mathcal{O}$.

Note that on a compact set $\mathcal{O} \subset R^n$, $|||D^2 f||| := \sup_{x\in\mathcal{O}} |||\frac{\partial^2 f}{\partial x^2}(x)|||$ and $|||D^2 h||| := \sup_{x\in\mathcal{O}} |||\frac{\partial^2 h}{\partial x^2}(x)|||$ are bounded, and Assumption 3.1.4 holds for all $x, y \in \mathcal{O}$. Let $\alpha_1 = \|N\|^2(1 + \|A\|^2 + \|A\|^4 + \cdots + \|A\|^{2(n-2)})$, $\alpha_2 =$ minimum eigenvalue of NN^T, $a = max(1, \|A\|)$, and

$$\begin{aligned} \beta_k &= (1 + \|\bar{P}_0\| \, \|Dh\|^2)a^k \prod_{l=1}^{k} \{1 + \|Dh\|^2 \\ &\quad \times [\|A\|^{2l}\|\bar{P}_0\| + \|N\|^2(\|A\|^{2(l-1)} + \|A\|^{2(l-2)} + \cdots + 1)]\}. \end{aligned}$$

Theorem 5.2 *Consider the system (3.1) and its associated EKF (3.3) and (3.4). Suppose that the system (3.1) satisfies the observability rank condition on a convex compact set $\mathcal{O}$, and that $[\frac{\partial f}{\partial x}(x)]^{-1}$ exists at each $x \in \mathcal{O}$. Let $\delta_1 > 0$ be a constant which satisfies the inequality (4.6) for some $0 < \gamma_1 \le \gamma_2$. Let $p = (\gamma_2 + 1/\alpha_2)$, $q = a^2(\alpha_1 + 1/\gamma_1) + \|N\|^2$. Let $\delta_2 > 0$ be such that $\varphi((pq)^{1/2}\delta_2, |||D^2 f|||, |||D^2 h|||) \le -\gamma$ for some $\gamma > 0$, where φ is defined in Section 3, and let M be the smallest integer which satisfies*

$$\begin{aligned} &[1 + (q\|A\|^2 + \|N\|^2)\|Dh\|^2]\|A\|(1 + q\|Dh\|^2) \\ &\times (1 - \tfrac{\gamma}{p})^{M/2}(pq)^{1/2} < 1. \end{aligned}$$

Suppose further that $x_k \in \mathcal{O}_\epsilon$, $k \ge 0$, for some $\epsilon > 0$, and that $|e_0| \le \frac{\delta}{\beta_{n+M-1}}$ with $\delta = min(\epsilon/2, \delta_1, \delta_2)$. Then we have the following results:

1. $|\bar{x}_k - x_k| \le \delta$ *and* $|\hat{x}_k - x_k| \le \delta \quad \forall k \ge 0$.

Recall that $R = I$ is used as a design variable. Also,

$$\begin{aligned}|\hat{x}_{n+M} - x_{n+M}| &\leq (1 + \|K_{n+M}\| \, \|Dh\|)|e_{n+M}| \\ &\leq [1 + (q\|A\|^2 + \|N\|^2)\|Dh\|^2]\|A\|(1 + q\|Dh\|^2) \\ &\quad \times (1 - \frac{\gamma}{p})^{M/2}(pq)^{1/2}|e_{n-1}| \leq \delta.\end{aligned}$$

In addition, we have

$$\bar{x}_{n+M} \in \mathcal{O}_{\epsilon/2} \quad \text{and} \quad \hat{x}_{n+M} \in \mathcal{O}_{\epsilon/2}.$$

Thus the conditions (5.1) and (5.2) are also met for $k = n + M$. Hence $\|P_{n+M}\| \leq \alpha_1 + 1/\gamma_1$ and $\|P_{n+M}^{-1}\| \leq p$. Therefore by induction it holds that for $k \geq n + M$,

1. $|\bar{x}_k - x_k| \leq \delta \leq \epsilon/2, \quad |\hat{x}_k - x_k| \leq \delta \leq \epsilon/2.$
2. $\|\bar{P}_k\| \leq q, \quad \|P_k^{-1}\| \leq p.$
3. $|e_k| \leq \delta(pq)^{1/2}(1 - \frac{\gamma}{p})^{(k-n+1)/2}.$ □

Remark 5.3

(a) In order to satisfy the observability condition, it is necessary to keep the estimates $\bar{x}_k$ and $\hat{x}_k$ near x_k for $0 \leq k \leq n+M-1$, thus requiring a good initial guess.

(b) We also need to have a converging period $(n-1 \leq k \leq n+M-1)$ for the EKF in order to build up the observability condition; after this, the recursions proceed automatically.

Remark 5.4 The above results hold wherever the initial guess is close enough to the true state. In other words, we have convergence of the observation error on an open neighborhood of the diagonal of the product space of the true state and the estimate, which includes the origin. This kind of observer is termed *quasi-local* [11]. Note that most results on local observers are only valid on an open neighborhood of the origin [8, 16, 17].

6 Conclusions

Motivated by the fact that the EKF can be used as a parameter estimator, we have analyzed in detail how the EKF works when it is used as an observer for general discrete-time nonlinear systems. First, we gave a new proof of the fact that the Kalman filter is a global observer for linear (discrete-time) time-varying systems. Based on this proof, we were able to

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