Supplemental material for “Control Barrier Function Based Quadratic Programs with Application to Automotive Safety Systems”

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This document develops two functions, h_F^c and h_F^o, that can be used to define the safe set $\mathcal{C}$ in Section V.A. of the paper.

Recall that the dynamics of the adaptive cruise control (ACC) model can be compactly expressed as

$$\dot{x} = \begin{pmatrix} -F_r/M \\ a_L \\ x_2 - x_1 \end{pmatrix} + \begin{pmatrix} 1/M \\ 0 \\ 0 \end{pmatrix} u. \tag{1}$$

If we drop the aerodynamic drag term F_r in (1), then an reciprocal control barrier function (RCBF) B_F meeting the comfort constraint

$$u \leq a_f' Mg, \quad -u \leq a_f Mg. \tag{FC}$$

for the simplified dynamic is also a RCBF for the original model (1), because the drag terms effectively augment the action of braking. In what follows, we develop the RCBFs for the ACC problem with input constraints using the simplified model, and the *optimality* is in the sense of the simplified model.

For simplicity, suppose that the present time is $t_0 = 0$ and denote $v_f(0), v_l(0)$ and $D(0)$ by v_f, v_l, and D respectively. If the lead car brakes using its maximal deceleration force, then the best response of the controlled car is to brake using its maximal deceleration force. Let T_l and T_f denote the time in seconds for the two cars to come to a complete stop when using maximum braking force, respectively. Supposing there are no reaction delays, it follows that $T_l = \frac{v_l}{a_l g}, T_f = \frac{v_f}{a_f g}$.

I. Optimal CBF for ACC

The *optimal RCBF* takes the form of $B_F^o = \frac{1}{h_F^o}$ or $B_F^o = -log(\frac{h_F^o}{1+h_F^o})$ with $h_F^o(x) = D - \Delta^*$ where Δ^* is given as follows:
(i) if $T_f > T_l$, that is, the lead car can stop first, then

$$\Delta^* = \max_{t \in [0, T_f]} (\Delta_1(t) + \tau_d(v_f - a_f g t)).$$

with

$$\Delta_1(t) = \begin{cases} (v_f t - \frac{1}{2} a_f g t^2) - (v_l t - \frac{1}{2} a_l g t^2), t \in [0, T_l), \\ (v_f t - \frac{1}{2} a_f g t^2) - \frac{v_l^2}{2 a_l g}, t \in [T_l, T_f]. \end{cases}$$

(ii) if $T_f \leq T_l$, that is, the following car can stop first, then

$$\Delta^* = \max_{t \in [0, T_f]} (\Delta_2(t) + \tau_d(v_f - a_f g t)),$$

with

$$\Delta_2(t) = (v_f t - \frac{1}{2} a_f g t^2) - (v_l t - \frac{1}{2} a_l g t^2),\ t \in [0, T_f],$$

For both cases, $\Delta_1(t)$ or $\Delta_2(t)$ is the *decrease in relative distance* at time t between the two cars, and Δ^* is the maximum decrease in distance headway.

The explicit form of Δ^* can be derived by solving the optimization problem above. The results are as follows:

Case(I) $a_f = a_l$

(Ia)$v_f \leq v_l$:

$$\Delta^* = 1.8v_f$$

(Ib)$v_f > v_l$

(Ib-1)$v_l < v_f \leq v_l + 1.8a_f g$:

$$\Delta^* = 1.8v_f$$

(Ib-2)$v_f > v_l + 1.8a_f$:

$$\Delta^* = \frac{1}{2}\frac{(1.8a_f g - v_f)^2}{a_f g} + 1.8v_f - \frac{v_l^2}{2a_l g}$$

Case(II) $a_f > a_l$

(IIa) $v_f < \frac{a_f}{a_l}v_l$

(IIa-1) $v_l \geq v_f$:

$$\Delta^* = 1.8v_f$$

(IIa-2) $v_l < v_f$

(IIa-2-1) $v_f \leq v_l + 1.8a_f g$:

$$\Delta^* = 1.8v_f$$

(IIa-2-2) $v_f > v_l + 1.8a_f g$:

$$\Delta^* = \frac{1}{2}\frac{(v_l + 1.8a_f g - v_f)^2}{(a_f - a_l)g} + 1.8v_f$$

(IIb) $v_f \geq \frac{a_f}{a_l}v_l$

(IIb-1) $v_f \leq v_l + 1.8a_f g$:

$$\Delta^* = 1.8v_f$$

(IIb-2) $v_f > v_l + 1.8a_f g$

(IIb-2-1) $\frac{v_f - 1.8a_f g}{a_f g} \leq \frac{v_l}{a_l g}$:

$$\Delta^* = \frac{1}{2}\frac{(v_l + 1.8a_f g - v_f)^2}{(a_f - a_l)g} + 1.8v_f$$

(IIb-2-2) $\frac{v_f - 1.8a_f g}{a_f g} > \frac{v_l}{a_l g}$:

$$\Delta^* = \frac{1}{2}\frac{(1.8a_f g - v_f)^2}{a_f g} + 1.8v_f - \frac{v_l^2}{2a_l g}$$

Case(III) $a_f < a_l$

(IIIa) $v_f \leq \frac{a_f}{a_l} v_l$:

$$\Delta^* = 1.8v_f$$

(IIIb) $v_f > \frac{a_f}{a_l} v_l$

(IIIb-1) $v_f \leq v_l$

(IIIb-1-1) $v_f \leq 1.8a_f g + \frac{a_f}{a_l} v_l$:

$$\Delta^* = 1.8v_f$$

(IIIb-1-2) $v_f > 1.8a_f g + \frac{a_f}{a_l} v_l$:

If $(v_l - v_f + 1.8a_f g)^2 \geq \frac{[a_l(v_f - 1.8a_f g) - a_f v_l]^2}{a_l a_f}$,

$$\Delta^* = 1.8v_f$$

If $(v_l - v_f + 1.8a_f g)^2 < \frac{[a_l(v_f - 1.8a_f g) - a_f v_l]^2}{a_l a_f}$,

$$\Delta^* = \frac{1}{2}\frac{(1.8a_f g - v_f)^2}{a_f g} + 1.8v_f - \frac{v_l^2}{2a_l g}$$

(IIIb-2) $v_f > v_l$

(IIIb-2-1) $v_f \geq 1.8a_f g + v_l$:

$$\Delta^* = \frac{1}{2}\frac{(1.8a_f g - v_f)^2}{a_f g} + 1.8v_f - \frac{v_l^2}{2a_l g}$$

(IIIb-2-2) $v_f < 1.8a_f g + v_l$

(IIIb-2-2-1) $v_f \leq 1.8a_f g + \frac{a_f}{a_l} v_l$:

$$\Delta^* = 1.8v_f$$

(IIIb-2-2-2) $v_f > 1.8a_f g + \frac{a_f}{a_l} v_l$

If $(v_l - v_f + 1.8a_f g)^2 \geq \frac{[a_l(v_f - 1.8a_f g) - a_f v_l]^2}{a_l a_f}$,

$$\Delta^* = 1.8v_f$$

If $(v_l - v_f + 1.8a_f g)^2 < \frac{[a_l(v_f - 1.8a_f g) - a_f v_l]^2}{a_l a_f}$,

$$\Delta^* = \frac{1}{2}\frac{(1.8a_f g - v_f)^2}{a_f g} + 1.8v_f - \frac{v_l^2}{2a_l g}$$

II. Convervative CBF for ACC

A simpler, albeit conservative, estimate of the safe set is given in this section. Replace the term $\tau_d(v_f - a_f g t)$ by $\tau_d v_f$ in Δ^* and denote the new formula by Δ^{c*}. Then we define $h_F^c(x) = D - \Delta^{c*}$ and call the associated RCBF candidate B_F^c as in $B_F^c := 1/h_F^c$ or $B_F^c = \log(\frac{h_F^c}{1+h_F^c})$ the *conservative RCBF*. Clearly, Δ^{c*} is larger than Δ^* and therefore h_F^c corresponds to a more conservative safety set than h_F^o. The closed-form expressions for h_F^c are much simpler than those for h_F^o and are given below:

Case (i). $v_l \geq v_f, T_l \geq T_f$:

$$h_F^c(x) = D - \tau_d v_f.$$

Case (ii). $v_l \geq v_f, T_l < T_f$:

$$h_F^c(x) = D - \tau_d v_f - \frac{1}{2}\frac{(a_l v_f - a_f v_l)^2}{a_l a_f (a_l - a_f) g}.$$

Case (iii). $v_l < v_f, T_l \geq T_f$:

$$h_F^c(x) = D - \tau_d v_f - \frac{1}{2}\frac{(v_f - v_l)^2}{(a_f - a_l) g}.$$

Case (iv). $v_l < v_f, T_l < T_f$:

$$h_F^c(x) = D - \tau_d v_f - \frac{1}{2}\frac{v_f^2 a_l - v_l^2 a_f}{a_f a_l g}.$$

Note that in Case (ii), $v_l \geq v_f$ and $T_l < T_f$ imply that $a_f \neq a_l$. Similar reasoning applies in Case (iii).

III. Proof of Validity of CBFs

First, we prove that the RCBF candidate $B_F^c := 1/h_F^c$ satisfies

$$\inf_{u \in U}\left[L_f B_F^c + L_g B_F^c u - \frac{1}{B_F^c}\right] \leq 0, \tag{2}$$

where h_F^c is given above and u satisfying (FC):

Case (i). $v_l \geq v_f, T_l \geq T_f$:

$$h_F^c(x) = D - \tau_d v_f.$$

Because

$$\dot{B}_F^c = -\frac{-\tau_d u/M + (v_l - v_f)}{(h_F^c)^2}.$$

If $u = -a_f M g$, then

$$L_f B_F^c + L_g B_F^c u = -\frac{\tau_d a_f g + (v_l - v_f)}{(h_F^c)^2} < 0,$$

which means that (2) holds.

Case (ii). $v_l \geq v_f, T_l < T_f$:

$$h_F^c(x) = D - \tau_d v_f - \frac{1}{2}\frac{(a_l v_f - a_f v_l)^2}{a_l a_f (a_l - a_f) g}.$$

Because

$$\dot{B}_F^c = -\frac{-\tau_d u/M + (v_l - v_f) - \frac{(a_l v_f - a_f v_l)(u a_l/M - a_f a_L)}{(a_l a_f (a_l - a_f) g)}}{(h_F^c)^2}.$$

If $u = -a_f M g$, then

$$L_f B_F^c + L_g B_F^c u = -\frac{\tau_d a_f g + (v_l - v_f) + \frac{(a_l v_f - a_f v_l)(a_l a_f g + a_f a_L)}{a_l a_f (a_l - a_f) g}}{(h_F^c)^2} < 0,$$

because $a_l > a_f$ and $a_l v_f > a_f v_l$ in this case. Then it follows that inequality (2) holds.

Case (iii). $v_l < v_f, T_l \geq T_f$:

$$h_F^c(x) = D - \tau_d v_f - \frac{1}{2}\frac{(v_f - v_l)^2}{(a_f - a_l) g}.$$

Because

$$\dot{B}_F^c = -\frac{-\tau_d u/M + (v_l - v_f) - \frac{(v_f - v_l)(u/M - a_L)}{(a_f - a_l) g}}{(h_F^c)^2}.$$

If $u = -a_f M g$, then

$$L_f B_F^c + L_g B_F^c u = -\frac{\tau_d a_f g + \frac{(v_f - v_l)(a_l g + a_L)}{(a_f - a_l) g}}{(h_F^c)^2} < 0,$$

because $v_f > v_l$ and $a_f > a_l$ in this case. It follows that inequality (2) holds.

Case (iv). $v_l < v_f, T_l < T_f$:

$$h_F^c(x) = D - \tau_d v_f - \frac{1}{2}\frac{v_f^2 a_l - v_l^2 a_f}{a_f a_l g}.$$

Because

$$\dot{B}_F^c = -\frac{-\tau_d u/M + (v_l - v_f) - \frac{(a_l v_f u/M - a_f v_l a_L)}{a_f a_l g}}{(h_F^c)^2}.$$

If $u = -a_f M g$, then

$$L_f B_F^c + L_g B_F^c u = -\frac{\tau_d a_f g + v_l + v_l a_L / a_l g}{(h_F^c)^2} < 0,$$

which means that inequality (2) holds.

On the other hand, if B_F^c is taken as

$$B_F^c = \log\left(\frac{h_F^c}{1 + h_F^c}\right)$$

then because $\dot{B}_F^c = -\dot{h}_F^c / (h_F^c(1 + h_F^c))$, it is easy to check that argument above is still valid for each case after minor modifications. Similarly, the optimal RCBF candidate B_F^o associated with h_F^o can also be shown to be a valid RCBF for the ACC problem.