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Sampling, Infinite Zeros and Decoupling of Linear Systems*†

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An analysis of the infinite zero structure of a sampled-data system leads to a determination of whether noninteracting control, left(right)-invertibility and disturbance decouplability are preserved under time-sampling.

Key Words—Decoupling; disturbance rejection; inverse systems; sampled-data systems; infinite zeros.

Abstract—In order to understand more fully some of the trade-offs involved in using a sampled-data representation of a continuous-time system, the effects of time-sampling on the ability to achieve disturbance decoupling and input–output decoupling for linear systems are investigated. It is shown that disturbance decouplability is lost through sampling whereas row-by-row dynamic input–output decouplability is preserved in a very strong way. These results are obtained by analyzing the structure at infinity of a sampled-data system.

1. INTRODUCTION AND MOTIVATION

EVEN WHEN one is controlling a continuous-time system, prevailing technology usually dictates that the compensator be implemented digitally, and hence operated in discrete-time. Such an implementation motivates working directly with a sampled-data model of the plant. This may be especially true if one is employing some sort of identification scheme in order to obtain a model.

It is well known that if a finite-dimensional, continuous-time, time-invariant, linear system is controllable (observable) then, except possibly for a "small" set of sampling times, its sampled-data versions will also be controllable (observable) (Gibson and Ha, 1980; Chen, 1984). This guarantees that if pole-placement, observer design, or optimal quadratic control are viable controller design methodologies for a continuous-time system, then they will also be applicable to a sampled-data representation of the plant. In other words, with respect to the above methodologies, no design possibilities are lost through sampling.

However, if the applicability of a given design strategy depends upon something more than just controllability or observability, if it exploits some particular structural aspect of the system, for example, its inherent integration structure [i.e. the number of integrators (or delays) separating sensors from actuators], then a sampled-data model of a system may not be as suitable for design purposes as a continuous-time model. In particular, the conditions for applicability of the method in question may not be satisfied, and hence one is better off basing the design upon a continuous-time model and then implementing it digitally via rapid sampling, using the generalized sampled-data hold techniques of Kabamba (1987), or seeking an approximate solution via the almost-invariant methods of Willems (1981).

The above discussion is intended to motivate a more in-depth study of the effects of time-sampling on a linear system. This paper (see also Shor, 1987) will be concerned with the following questions. Suppose that for a given continuous-time linear system disturbance decoupling or noninteracting control is achievable, will it also be achievable for sampled-data versions of the system? If a continuous-time system is left- or right-invertible, will sampled-data representations of the system also be? Though not investigated here, one could also consider model matching (Moore and Silverman, 1972).

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2. SOME SYNTHESIS PROBLEMS

Consider a linear system

$$\Sigma: \begin{aligned} \dot{x} &= Ax + Bu + Dw \\ y &= Cx \end{aligned} \qquad (2.1)$$

whose inputs have been separated into controls $u \in \mathbf{R}^m$ and disturbances $w \in \mathbf{R}^d$, and with state $x \in \mathbf{R}^n$ and output $y \in \mathbf{R}^p$. Suppose that the controls are applied in a piecewise constant fashion so that u is constant over the half-open intervals $[kT, (k+1)T)$, and moreover, suppose that it is reasonable to approximate† (or model) the disturbance w as also being piecewise constant. Then letting x_k, u_k, w_k and y_k denote $x(kT)$, $u(kT)$, $w(kT)$ and $y(kT)$, respectively, one obtains a sampled-data representation of Σ

$$\Sigma_s(T): \begin{aligned} x_{k+1} &= \bar{A}x_k + \bar{B}u_k + \bar{D}w_k \\ y_k &= Cx_k \end{aligned} \qquad (2.2)$$

where $\bar{A} = \exp(AT)$, $\bar{B} = \int_0^T \exp(A\tau)B\,\mathrm{d}\tau$ and $\bar{D} = \int_0^T \exp(A\tau)D\,\mathrm{d}\tau$. Note that C is unchanged. Sometimes, the notation $\bar{A}(T)$, $\bar{B}(T)$, $\bar{D}(T)$ will be used to emphasize their dependence on the sampling time.

Several synthesis problems and their conditions for solvability are now summarized for continuous-time systems. The case of discrete-time systems is exactly parallel.

2.1. *Disturbance decoupling*

A system is said to be *disturbance decouplable* (Wonham, 1979) if one can find a state variable feedback‡ $u = Fx + v$ (in continuous- or discrete-time) so that the resulting closed-loop system is disturbance decoupled; that is, the disturbances do not affect the output. Letting V^* denote the maximal controlled-invariant subspace contained in $\ker C$, it is known that Σ can be disturbance decoupled if and only if $\operatorname{Im} D \subset V^*$. This will be exploited when the effects of time sampling on disturbance decouplability are considered.

2.2. *Invertibility*

Throughout the rest of this section, it is assumed that the matrix D in (2.1) is identically zero; i.e. there are no external disturbances acting on the system.

The linear system Σ is said to be left (right)-invertible if its transfer function

$$G(s) = C(sI - A)^{-1}B$$

is left (right)-invertible over the field of rational functions in s (Silverman, 1969; Sain and Massey, 1969; Hautus and Silverman, 1983; Chen, 1984). In particular, if the rank of $G(s)$ equals the number of inputs, the system is left-invertible, and if the rank of $G(s)$ equals the number of outputs, the system is right-invertible. Roughly speaking, left-invertibility concerns the ability to recover the input from the output and its derivatives, and right-invertibility concerns the ability to produce (or track) a very rich set of output functions via appropriately applied controls. Right-invertibility is known to play an important role in the input–output decoupling problem (Morse and Wonham, 1971).

† In particular, this is exact for biases and step disturbances.

‡ It can be shown that dynamic state variable feedback does not enlarge the class of disturbance decouplable systems.

2.3. *Input–output decoupling*

Suppose that the output of the linear system (2.1) has been partitioned into blocks; that is, $y' = (y^{1\prime}, \ldots, y^{\nu\prime})$ ($'$ denotes transpose) and each y^i is (possibly) a vector of outputs. One then says that (2.1) is *regularly statically input–output decouplable* (Wonham, 1979; Morse and Wonham, 1971) if there exists a regular§ static state-variable feedback $u = Fx + Gv$ and a block partitioning of the new inputs $v' = (v^{1\prime}, \ldots, v^{\nu\prime})$ with respect to which the resulting closed-loop system is input–output decoupled; that is, v^i does not affect y^j for $j \neq i$. When each y^i is a scalar, this is referred to as *row-by-row decoupling* as opposed to *block decoupling* when at least one y^i has dimension greater than one.

In contrast to disturbance decoupling, quite a bit can be gained by allowing dynamic compensation in order to achieve input–output decoupling (Morse and Wonham, 1971; Wonham, 1979, Chapter 9). A system is said to be *dynamically input–output decouplable* if it can be decoupled with regular dynamic state-variable feedback, that is, a feedback of the form

$$\begin{aligned} u &= F_1x + F_2z + G_1v \\ \dot{z} &= H_1z + H_2x + G_2v. \end{aligned}$$

where the transfer function from v to u is right-invertible. When z is zero-dimensional this reduces to regular static state-variable feedback.

2.4. *Zeros at infinity*

The properties of left- or right-invertibility and static or dynamic input–output decouplability can be characterized in terms of the system's so-called *structure at infinity* (Commault and Dion, 1982; Rosenbrock, 1970; Morse, 1976; Verghese, 1978; Pugh and Ratcliff, 1979; Silverman and Kitapci, 1983).

§ Regular means that $|G| \neq 0$. This is to avoid certain difficulties; see Descusse *et al.* (1985).

The currently accepted definition proceeds as follows. A matrix of transfer functions $M(s)$ is said to be *bicausal* if $M(s)$ is proper and $M^{-1}(s)$ exists and is proper. Then, given any transfer matrix $G(s)$, there exist bicausal matrices $M(s)$ and $N(s)$, and unique integers $\{m_1, \ldots, m_k\}$ such that

$$G(s) = M(s)\,\mathrm{diag}\left(\frac{1}{s^{m_1}}, \ldots, \frac{1}{s^{m_k}}, 0, \ldots, 0\right)N(s).$$

One says that $G(s)$ has k *zeros at infinity* of *orders* $\{m_1, \ldots, m_k\}$. It is often convenient to define $p^\mu :=$ cardinality $\{m_i \mid m_i \geq \mu\}$, the number of zeros at infinity of order greater than or equal to μ. Note that p^1, the number of zeros at infinity, is equal to the rank of $G(s)$ over the field of rational functions of s, $\mathbf{R}(s)$.

Some recent work by Malabre (1982) and Nijmeijer and Schumacher (1985) shows how to calculate the structure at infinity of a linear system

$$\Sigma: \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

from the V^*-algorithm (Wonham, 1979). One first defines

$$\begin{aligned} &V^0 = \mathbf{R}^n \\ &V^{\mu+1} = \ker C \cap A^{-1}(V^\mu + \operatorname{Im} B) \quad \mu \geq 0. \end{aligned} \tag{2.3}$$

It can easily be shown that this algorithm converges in at most n steps to V^*, the maximal controlled-invariant subspace contained in the kernel of C. One can calculate the list of integers $\{p^\mu\}_{\mu=1}^n$ by

$$p^\mu = \dim(\operatorname{Im} B \cap V^{\mu-1}) - \dim(\operatorname{Im} B \cap V^*). \tag{2.4}$$

The same procedure can be applied to the "subsystem"

$$\Sigma^i: \begin{aligned} \dot{x} &= Ax + Bu \\ y^i &= C^i x \end{aligned}$$

to obtain, for each i, the corresponding list $\{p_i^\mu\}_{\mu=1}^n$. Finally, applying the above to the sampled system $\Sigma_S(T)$, and to the sampled "subsystems", $\Sigma_S^i(T)$, one obtains the lists $\{\bar{p}^\mu(T)\}_{\mu=1}^n$ and $\{\bar{p}_i^\mu(T)\}_{\mu=1}^n$, $i = 1, \ldots, v$.

The importance of these integers is given in the following Lemma.

Lemma 2.1.

(a) Σ is left-invertible if, and only if, $p^1 = m$, the number of inputs (Silverman, 1969; Malabre, 1982).

(b) Σ is right-invertible if, and only if, $p^1 = p$, the number of scalar output components (Silverman, 1969; Malabre, 1982).

(c) Σ is regularly statistical input–output decouplable† if, and only if (Descusse *et al.*, 1983),

$$\sum_{i=1}^{v} p_i^\mu = p^\mu \quad \mu = 1, \ldots, n.$$

(d) Σ is dynamically input–output decouplable if, and only if (Descusse, 1987; see also Appendix B),

$$\sum_{i=1}^{v} p_i^1 = p^1.$$

The same results hold for discrete-time systems. Hence, to understand how the invertibility and input–output decouplability properties of a system are affected by time sampling, it suffices to analyze the effects of sampling on the system's structure at infinity. In Åström *et al.* (1984) the effects of sampling on the finite zeros of a single-input single-output system are investigated.

3. SAMPLING, INFINITE ZEROS, AND SYNTHESIS PROBLEMS

Recall, from the previous section, that the conditions for left- and right-invertibility and dynamic input–output decouplability depend only on the number of zeros at infinity of the system and associated subsystems, whereas the condition for static input–output decouplability also depends on the orders of the zeros at infinity.

3.1. *Number of infinite zeros, invertibility and dynamic I–O decoupling*

The effect of time sampling on the number of zeros at infinity of a continuous-time linear system is as follows‡.

Theorem 3.1. Suppose that the continuous-time system Σ has $p^1 = \min\{m, p\}$ zeros at infinity. Then, for all T in the complement of a discrete set with no accumulation points, the associated sampled system $\Sigma_s(T)$ also has $\bar{p}^1(T) = \min\{m, p\}$ zeros at infinity [i.e. for any bounded subset of the reals, the subset of sampling times for which $\bar{p}^1(T) < p^1$ is finite].

Remark 3.2. As noted by one of the reviewers, the above may even hold for all $T \neq 0$.

The consequence of this for preservation of left- and right-invertibility under sampling is Corollary 3.3.

Corollary 3.3. Suppose that the continuous-time system Σ is left (right)-invertible. Then the

† It is emphasized that decoupling with internal stability requires additional considerations.

‡ Proofs of the theorems are given in subsequent sections.

sampled-data system $\Sigma_s(T)$ is left (right)-invertible for all sampling times T in the complement of a discrete set with no accumulation points.

Hence, invertibility of a linear system is essentially preserved under the operation of time-sampling. Therefore, any design scheme depending on the invertibility properties of the system (for example, tracking-type problems) would be just as applicable to the sampled system (2.2) as it would have been to the original continuous-time system (2.1). In this respect, one has not lost anything by passing directly to a sampled-data representation of the system.

The consequence of Theorem 3.1 for the preservation of row-by-row dynamic input–output decouplability is Corollary 3.4.

Corollary 3.4. Suppose that Σ can be row-by-row decoupled with dynamic feedback. Then $\Sigma_s(T)$ can also be row-by-row decoupled with dynamic feedback for all T in the complement of a discrete set without accumulation points.

Hence, with regards to dynamic *row-by-row* decoupling, no design possibilities are lost through the introduction of time-sampling.

Note that a key assumption in Theorem 3.1 is that the underlying continuous-time system has the maximum possible number of zeros at infinity. If the continuous-time system has fewer zeros at infinity, then sampling may indeed increase the number of zeros at infinity. Consequently, a system may become invertible through the introduction of sampling, and dynamic *block* input–output decouplability may be destroyed.

To see this, consider a system with transfer function

$$G(s)=\begin{bmatrix}\dfrac{1}{s+1} & \dfrac{1}{s+2}\\ \dfrac{1}{s(s+1)} & \dfrac{1}{s(s+2)}\end{bmatrix},$$

which clearly has rank one over the field of rational functions in s, hence, one zero at infinity. One calculates the sampled system $G_T(z)$ to be

$$G_T^{11}(z)=\frac{1-e^{-T}}{z-e^{-T}}\quad G_T^{12}(z)=\frac{1}{2}\frac{1-e^{-2T}}{z-e^{-2T}}$$

$$G_T^{21}=\frac{1}{z-1}\left[\frac{(1-e^{-T})^2}{z-e^{-T}}+(T+e^{-T}-1)\right]$$

$$G_T^{22}(z)=\frac{1}{z-1}\left[\frac{1}{4}\frac{(1-e^{-2T})^2}{z-e^{-2T}}\right.$$
$$\left.+\left(\frac{1}{2}T+\frac{1}{4}e^{-2T}-\frac{1}{4}\right)\right],$$

which, for all $T\neq 0$, has two zeros at infinity and is *invertible* over the field of rational functions in z.

From this observation, it is easy to exhibit continuous-time systems that are dynamically block input–output decouplable, but for which this property is destroyed by time sampling. Simply consider

$$\tilde{G}(s)=\begin{bmatrix}\dfrac{1}{s+1} & \dfrac{1}{s+2}\\ \dfrac{1}{s(s+1)} & \dfrac{1}{s(s+2)}\\ \dfrac{1}{s+3} & \dfrac{1}{s+4}\end{bmatrix}\tag{3.1}$$

and let the output blocks be given by the first two rows, and the last row, of $G(s)$, respectively. Then one calculates that

$$p^1=2,\quad p_1^1=1,\quad p_1^1=1,$$

establishing the dynamic block input–output decouplability of (3.1), whereas (for $T\neq 0$)

$$\bar{p}^1(T)=2,\quad \bar{p}_1^1(T)=2,\quad \bar{p}_2^1(T)=1,$$

showing that $\tilde{G}_T(z)$ *cannot* be dynamically input–output decoupled.

3.2. *Orders of infinite zeros and static I–O decoupling*

Attention is now turned to static row-by-row decoupling, that is, to row-by-row input–output decoupling with static state variable feedback. According to Lemma 2.1, one must study how the *orders* of the zeros at infinity are affected by sampling; in the case of dynamic decoupling, it was sufficient to understand how their *number* was affected.

When Σ is a (strictly proper) SISO system and its transfer function is nonzero, it is well-known that $\Sigma_S(T)$ will have one zero at infinity of order 1, for "almost all" sampling times T, irrespective of the relative degree of Σ. Indeed, the rank of $c(T)b(T)$, the first Markov parameter, will be one for "almost all" T.

A natural MIMO conjecture would be: suppose that Σ has $p^1=\min\{m,p\}$ zeros at infinity, then $\Sigma_S(T)$ also has p^1 zeros at infinity, all of order 1, for "almost all" sampling times T. The following example shows that this need not be the case. Consider the linear system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}0&0&0&0\\1&0&3&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}+\begin{bmatrix}1&0\\0&0\\0&0\\0&1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}$$

$$y=\begin{bmatrix}1&0&2&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}.$$

One calculates that it has two zeros at infinity, of orders 1 and 3. However, a straightforward computation gives that the sampled system has, for all $T > 0$, two zeros at infinity, one of order 1 and one of order 2.

Nevertheless, upon checking other examples, one gets the feeling that the conjecture is almost true. This is now made precise. Let $\mathbf{S}(n, m, p)$ be the set of all linear systems with an n-dimensional state, m inputs, p outputs, and $p^1 = \min\{m, p\}$ zeros at infinity. $\mathbf{S}(n, m, p)$ can be viewed as a subset of $\mathbf{R}^{n\times n} \times \mathbf{R}^{m\times n} \times \mathbf{R}^{p\times n}$; equip it with the induced Euclidean topology. [Since two systems differing only by a nonsingular state coordinate transformation are usually considered equivalent, one may wish to form the quotient of $\mathbf{S}(n, m, p)$ with $Gl(n)$, the general linear group, endowing $\mathbf{S}(n, m, p)/Gl(n)$ with the quotient topology. However, as the properties to be considered are invariant under $Gl(n)$, the topological statements which follow about $\mathbf{S}(n, m, p)$ are equally valid for $\mathbf{S}(n, m, p)/Gl(n)$.] A subset S of $\mathbf{S}$ is said to be *closed under feedback* if $(A, B, C) \in S$ implies that $(A + BF, B, C) \in S$ for all $n \times m$ matrices F.

Interesting examples of sets that are closed under feedback include: (a) the set of all systems with a given structure at infinity $\{p^\mu\}_{\mu=1}^{n}$; (b) the set of all systems with a given set of controllability indices; (c) the set of all systems feedback equivalent to a given system; and (d) unions and intersections of the above sets.

Theorem 3.5. Every set $S \subset \mathbf{S}(n, m, p)$ that is closed under feedback contains a relatively open and dense subset† S' to which is associated an integer p^1, such that $\Sigma \in S'$ implies that $\Sigma_S(T)$ has p^1 zeros at infinity, all of order 1, for all T in the complement of a discrete set with no accumulation points.

Roughly speaking, Theorem 3.5 asserts that even if the continuous-time system has a "rich" structure at infinity, "almost always" its sampled versions will have a *trivial* structure at infinity; namely, all the zeros at infinity will be of order one. The consequences of this are Corollary 3.6.

Corollary 3.6. Every $S \subset \mathbf{S}(n, m, p)$ that is closed under feedback contains a relatively open and dense subset S' on which the following statements are equivalent.

(a) Σ is dynamically row-by-row decouplable.‡

(b) $\Sigma_S(T)$ is dynamically row-by-row decouplable for all T in the complement of a discrete set without accumulation points [i.e. for any bounded subset of the reals, the subset of times at which $\Sigma_S(T)$ is *not* dynamically input–output decouplable is either empty or finite].

(c) $\Sigma_S(T)$ is regularly statically row-by-row decouplable for all T in the complement of a discrete set without accumulation points.

In other words, within the class of systems for which input–output decouplability of any form is preserved, those sampled-data systems which really require dynamic compensation in order to achieve decoupling are quite exceptional. In particular, one can take S to be the set of all continuous-time systems in $\mathbf{S}(n, m, p)$ that can be dynamically input–output decoupled but for which there does *not* exist any regular static feedback achieving decoupling. The theorem then asserts that, with the exception of a relatively closed and nowhere dense subset of S, the sampled-data representations of such systems will be input–output decouplable with regular static feedback.

3.3. *Disturbance decoupling*

Attention is now turned to the effects of sampling on the disturbance decouplability of a system.

Theorem 3.7. There exists an open interval of sampling times T for which the sampled-data system $\Sigma_S(T)$ is disturbance decouplable if, and only if, the underlying continuous-time system is already disturbance decoupled.

In other words, disturbance decouplability is destroyed by time-sampling, and therefore, if one wishes to design such a controller, it is imperative to work with a continuous-time representation of the system. It is perhaps interesting to note that if one obtains an *approximate* time-discretized system by applying an Euler integration scheme§ to (2.1), viz.

$$\hat{\Sigma}_E(T)\colon \begin{array}{l} \hat{x}_{k+1} = \hat{x}_k + TA\hat{x}_k + TBu_k + TDw_k \\ \hat{y}_k = C\hat{x}_k \end{array} \tag{3.2}$$

then $\hat{\Sigma}_E(T)$ is disturbance decouplable if, and only if, Σ is. As (3.2) can also be viewed as the first terms of a Taylor expansion (in T) of (2.2), this shows that the obstruction to disturbance decouplability is second-order, or higher, in the sampling interval T. On the other hand, it is also

† A stronger statement can be made by using the notion of algebraic sets.

‡ Here, static decoupling can be viewed as a special case of dynamic decoupling—the compensator having zero dynamic order.

§ (a) One should note that applying an Euler scheme to a linear system is a coordinate-free notion in contradistinction to the case of general nonlinear systems. (b) Such first order discretization schemes are only accurate for relatively small sampling intervals.

clear that $\hat{\Sigma}_E(T)$ is controllable (observable) if, and only if, Σ is, and thus loss of controllability (observability) is also a second-order, or higher, effect in the sampling interval.

Finally, instead of a piecewise constant approximation to w, suppose that one chooses to do a (continuous) piecewise linear approximation. This is equivalent to defining $\dot{w} = \alpha$, where α is piecewise constant. Since augmenting (2.1) with $\dot{w} = \alpha$ still results in a system of the form (2.1), Theorem 3.7 remains applicable and therefore the resulting sampled-data system will also not be disturbance decouplable for most sampling times T.

4. PROOFS OF THE MAIN RESULTS

In this section, proofs of the results announced in the previous section are provided.

4.1. *Proof of Theorem* 3.1.

Theorem 3.1 will be established by showing that $\bar{p}^1(T) \geq p^1$ for "almost all" $T \in \mathbf{R}$. Associated with the linear system (2.1), define a sequence of matrices by

$$J_k := \begin{bmatrix} CB & 0 & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ CA^{k-1}B & CA^{k-2}B & \cdots & CB \end{bmatrix} \qquad (4.1)$$

and define $R_k := \operatorname{rank} J_k$. Then it is known (Sain and Massey 1969; Nijmeijer and Schumacher, 1985) that R_k is related to the system's structure at infinity per

$$R_k = kp^1 - kp^2 - \cdots - p^{k+1}$$

where, once again, p^μ is the number of zeros at infinity of order greater than or equal to μ. In particular,

$$p^1 = R_n - R_{n-1}$$

since $p^\mu = 0$ for $\mu \geq n+1$.

Lemma 4.1. Consider the discrete-time system

$$\hat{\Sigma}(T): \quad \begin{aligned} x_{k+1} &= \frac{1}{T}(\bar{A}(T) - I)x_k + \frac{1}{T}\bar{B}(T) \\ y_k &= Cx_k. \end{aligned}$$

Let $\hat{p}(T)$ be the number of zeros at infinity of $\hat{\Sigma}(T)$. Then, for all $T \neq 0$, $\hat{p}^1(T) = \bar{p}^1(T)$.

Proof. Defining $\hat{R}_k(T)$ in the obvious way, one easily shows that $\hat{R}_k(T) = \bar{R}_k(T)$ for all $T \neq 0$ and $k \geq 1$, which establishes the result. Alternately, one has

$$\begin{aligned} \hat{p}^1(T) &= \operatorname{rank} C\left[zI - \frac{1}{T}(\bar{A}(T) - I)\right]^{-1} \frac{1}{T}\bar{B}(T) \\ &= \operatorname{rank} C[(Tz+1)I - \bar{A}(T)]^{-1}\bar{B}(T) \\ &= (\text{all } T \neq 0) \operatorname{rank} C[zI - \bar{A}(T)]^{-1}\bar{B}(T) \\ &= \bar{p}_1(T) \end{aligned}$$

where the above ranks are with respect to the field of rational functions in z. □

From the above lemma, one deduces that it is sufficient to show that $\hat{p}^1(T) \geq p^1$ for all T in the complement of a discrete set with no accumulation points. First, it will be shown that, for $k \geq 1$, $\hat{R}_k(T) \geq R_k$ for "most" T. This will be done with the following lemma, which is itself a simple consequence of real analyticity.†

Lemma 4.2 (*Rank Lemma*). Let $M(T)$ be a matrix, each of whose entries is a real analytic function of T. Then for all T in the complement of a discrete set with no accumulation points,

$$\operatorname{rank} M(T) \geq \operatorname{rank} M(0).$$

For definiteness, let $\hat{J}_k(T)$ be the matrix J (4.1) with (A, B, C) replaced by

$$\left(\frac{1}{T}[\bar{A}(T) - I], \frac{1}{T}\bar{B}(T), C\right),$$

and define $\hat{R}_k(T) := \operatorname{rank} \hat{J}_k(T)$. Then, since

$$\frac{1}{T}[\bar{A}(T) - I] \xrightarrow[T\to 0]{} A$$

$$\frac{1}{T}\bar{B}(T) \xrightarrow[T\to 0]{} B,$$

it follows that $\hat{J}_k(T) \xrightarrow[T\to 0]{} J$. Defining $\hat{J}_k(0) := J$ results in $\hat{J}_k(T)$ being an analytic function of T. Hence, by the Rank Lemma, $\hat{R}_k(T) \geq \hat{R}_k(0) = R_k$, for all T in the complement of a discrete set with no accumulation points. □

To complete the proof of Theorem 3.1, recall that $\hat{p}^\mu(T) = p^\mu = 0$, $\mu \geq n+1$. Hence,

$$p^1 = \lim_{k\to\infty} \frac{1}{k} R_k$$

and similarly for $\hat{p}^1(T)$. Therefore, $\hat{R}_k(T) \geq R_1$ implies that $\hat{p}^1(T) \geq p^1$, which, in conjunction with Lemma 4.1, yields the desired result.

4.2. *Proof of Theorem* 3.5

From (2.3) and (2.4), one deduces that, for T fixed, $\Sigma_S(T)$ has $p^1 = \min\{m, p\}$ zeros at

† Recall that the zeros of a nontrivial analytic function are isolated.

infinity, all of order 1, if and only if

$$\operatorname{rank} \int_0^T C \exp(A\tau) B \, d\tau = p^1. \tag{4.2}$$

Lemma 4.3. Given any continuous-time linear system Σ having $p^1 = \min\{m, p\}$ zeros at infinity, there exists a (feedback) matrix F such that

$$\operatorname{rank} \int_0^T C \exp[(A + BF)\tau] B \, d\tau = p^1 \tag{4.3}$$

for all $T \neq 0$.

Delaying the proof of the lemma for a moment, it is shown how this establishes Theorem 3.5. Define

$$S' = \left\{ \Sigma \in S \mid \operatorname{rank} \int_0^T C \exp(A\tau) = p^1 \quad \text{for an open set of } T \right\}.$$

S' is easily shown to be a relatively open subset of S. Even more, it is dense because (4.3) and real analyticity imply that there exists $\delta^* > 0$ such that

$$\operatorname{rank} \int_0^T C \exp[(A + \delta BF)\tau] B \, d\tau = p^1$$

for all $\delta \in (0, \delta^*]$ and an open set of T. That is, one can always perturb A by an arbitrarily small amount, *within* S, and achieve (4.2).

Proof of Lemma 4.3. It suffices to show that there exist nonsingular matrices P and G, and a matrix F such that

$$\int_0^T PC \exp[(A + BF)\tau] BG \, d\tau \tag{4.4}$$

has the required rank properties. This will be accomplished by an algorithm which is in some sense dual to Silverman's Structure algorithm (Silverman, 1969, 1970; Silverman and Payne, 1971; Kitapci and Silverman, 1984); it will be dual in the sense that the transformations are performed on the inputs instead of the outputs.

Proposition 4.4 (*dual structure algorithm*). Let $\Sigma(A, B, C)$ be an n-dimensional right-invertible linear system with structure at infinity $\{p^\mu\}_{\mu=1}^n$. Let r_i be the number of zeros at infinity of order i; that is, $r_i = p^i - p^{i+1}$ for $i = 1, \ldots, n-1$ and $r_n = p^n$. Let $i_1 < i_2 < \cdots < i_k$ be such that $\{r_{i_1}, \ldots, r_{i_k}\}$ is the list of nonzero r_is ($\{i_1, \ldots, i_k\}$ are the orders of the system's zeros at infinity). Then there exists a regular static feedback $u = Fx + Gv$ such that, after a possible re-ordering of the outputs,

$$\frac{d^{i_1} y_1}{dt^{i_1}} = v_1$$

$$\frac{d^{i_2} y_2}{dt^{i_2}} = v_2 + \sum_{j=0}^{i_2 - i_1} C_2 \tilde{A}^{(i_2-1-j)} \tilde{B}_1 \frac{d^j v_1}{dt^j} \tag{4.5}$$

$$\frac{d^{i_k} y_k}{dt^{i_k}} = v_k + \sum_{\alpha=1}^{k-1} \sum_{j=0}^{i_k - i_\alpha} C_k \tilde{A}^{(i_k-1-j)} \tilde{B}_\alpha \frac{d^j v_\alpha}{dt^j}$$

where $\tilde{A} = A + BF$, $\tilde{B} = BG$, $[\tilde{B}_1 | \cdots | \tilde{B}_k] = \tilde{B}$ and $y' = [y_1', \ldots, y_k']$.

The proof of this proposition is given in Appendix A.

Without loss of generality, it is supposed that the linear system Σ of Lemma 4.3 is right-invertible; for if it is not, one can always delete components of the output until the system is right-invertible and still has p^1 zeros at infinity. Now let F and G be as in Proposition 4.4 and let P be a permutation matrix on the outputs so that (4.5) holds. Then (4.5) yields that (4.4) is a lower triangular matrix with diagonal

$$\left| \frac{T^{i_1}}{i_1!} I_1, \ldots, \frac{T^{i_k}}{i_k!} I_k \right|,$$

where I_j is the r_{i_j} identity matrix. This establishes that it is nonsingular for all $T \neq 0$. □

4.3. *Proof of Corollary* 3.6

Since each output component y^i is a scalar, $p_i^1 \in \{0, 1\}$. If $p_i^1 = 0$, then $\bar{p}_i^1(T) = 0$ for all T since y^i is then unaffected by the inputs. If $p_i^1 = 1$, then, by Theorem 3.1, $\bar{p}_i^1(T) = 1$ for "almost all" T. Hence, for "almost all" T, $\sum_{i=1}^p p_i^1 = \sum_{i=1}^p \bar{p}_i^1(T)$. Let S' be the open and dense subset of S where, for "almost all" T, $\bar{p}^1(T) = p^1$ and $\bar{p}^\mu(T) = 0$, $\mu \geq 2$. Then on S', Lemma 2.1 yields that dynamic and static input–output decouplability are equivalent.

4.4. *Proof of Theorem* 3.7

Suppose that $\Sigma_S(T)$ is disturbance decouplable for some open interval of sampling times T. Then it follows (Wonham, 1979) that

$$C \int_0^T \exp(A\tau) D \, d\tau \tag{4.6}$$

equals zero for all T in the same interval. Since (4.6) is an analytic function of T, it must therefore be true that (4.6) is identically zero. But this is equivalent to (4.6), and all of its derivatives, vanishing at $T = 0$. This yields

$$\operatorname{Im} D + A \operatorname{Im} D + \cdots + A^{n-1} \operatorname{Im} D \subset \ker C,$$

which shows that Σ is already disturbance decoupled (Wonham, 1979).

Since the other direction is trivial, the proof is complete. □

5. CONCLUSIONS AND COMMENTS

In an effort to understand some of the trade-offs involved in using a sampled-data representation of a continuous-time system for control purposes, this paper has investigated whether or not time-sampling introduces inherent obstructions to the solvability of various linear synthesis problems. This was done by analyzing the effects of time-sampling on a system's structure at infinity, that is, its inherent integration structure.

First, it was shown that (except possibly for a small set of sampling times) a sampled-data representation of a continuous-time system has at least as many zeros at infinity as the underlying continuous-time system (i.e. the rank of its transfer function will be as least as large). Hence, left (right)-invertibility or row-by-row dynamic decouplability of a continuous-time system implies that of its sampled-data representations. Through an example, it was shown that time-sampling can increase the number of zeros at infinity, resulting in an invertible (decouplable) sampled-data representation of a noninvertible (nondecouplable) continuous-time system.

Next, the effect of time-sampling on the orders of a system's zeros at infinity was investigated. For a strictly proper single-input single-output system it is well known that, irrespective of the system's relative degree and for almost all sampling times, its sampled-data representations have one zero at infinity of order 1. In other words, even though the continuous-time system may have its input separated from its output by a chain of integrators, time-sampling introduces additional finite zeros so that the input of the sampled-data system appears at the output after only one delay.

The case of multivariable systems was showed to be "richer" by exemplifying a 2-input 2-output continuous-time system, having two zeros at infinity of orders 1 and 3, for which its sampled-data representations had two zeros at infinity of orders 1 and 2 for all positive sampling times. That is, for multivariable systems, it is not necessarily true that time-sampling always leads to a trivial structure at infinity (i.e. all zeros at infinity are of order 1). However, further analysis showed this example to be "exceptional". "Most" (see Section 3.2 for a more precise statement) multivariable sampled-data systems do have a trivial structure at infinity. Consequently, most row-by-row decouplable (dynamically or statically) continuous-time systems become, upon the introduction of sampling, decouplable with static feedback. While this may seem to be a very positive feature of sampling, one must note that as the sampling interval becomes small, the static decoupling-feedback, in some sense, wants to "explode" into a dynamic compensator. This phenomenon deserves further investigation.

Lastly, it was shown that disturbance decouplability is destroyed by time-sampling in the sense that disturbance decouplability of the sampled-data representations, for an open set of sampling times, is equivalent to the continuous-time system being already disturbance decoupled.

Throughout the paper, the term "generic" (Wonham, 1979) has been carefully avoided, since, as remarked by one of the reviewers, ". . . [genericity] is a rather dangerous concept to use. Sets that may look "thin" from a particular mathematical perspective may not be so thin for certain applications, and if a given system is sufficiently "near" an exception[al] set, then the non-generic situation may be more typical for the system's behavior than the generic situation". With this in mind, the results concerning a system's structure at infinity may be summarized as saying that a sampled-data representation of a system tends to be more "generic" than the original system. For example, consider the set of 2-input 2-output continuous-time systems having an n-dimensional state. The property of having a zero at infinity or order greater than or equal to 2 is certainly physically ubiquitous, but it is non-generic. Nevertheless, after the introduction of time-sampling, a relatively open and dense subset of these systems will behave generically and have only zeros at infinity of order 1. In other words, even though the interconnection structure of a particular physical system may give it a rich (non-generic) structure at infinity, time-sampling will tend to wash this out, resulting in a system with a trivial (generic) structure at infinity.

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APPENDIX A: PROOF OF PROPOSITION 4.4 (DUAL STRUCTURE ALGORITHM)

Let J_k be as in (4.1), define $R_k := \operatorname{rank} J_k$, and recall that $R_k = kp^1 - p^2 - \cdots - p^{k+1}$. Let $R_0 = 0$ and define $\Delta_{k+1} := R_{k+1} - R_k$, $k = 0, 1, 2, \ldots$. It then follows that

$$\Delta_{k+1} = \Delta_k + r_k. \tag{A.1}$$

From (4.1) and (A.1) one deduces that $CA^jB = 0$ for all $0 \le j \le i_1 - 2$; that is, one must differentiate the output i_1 times before the input appears. Doing so one has

$$\frac{d^{i_1}y}{dt^{i_1}} = CA^{i_1}x + CA^{i_1-1}Bu$$

and moreover, rank $CA^{i_1-1}B = r_{i_1}$, the number of zeros at infinity of order i_1. If necessary, re-order the components of y so that the first r_{i_1} rows of $CA^{i_1-1}B$ are linearly independent. Write $C' = [C_1', \hat{C}_2']$, where C_1 has r_{i_1} rows. Let $u = F_1x + G_1v$ be any regular static feedback such that, upon writing $v' = [v_1', v_2']$, v_1 having r_{i_1} components, one has

$$v_1 = C_1A^{i_1}x + C_1A^{i_1-1}Bu;$$

this is always possible because rank $C_1A^{i_1-1}B = r_{i_1}$. Hence, for the resulting closed-loop systems one has

$$\frac{d^{i_1}y_1}{dt^{i_1}} = v_1$$

$$\frac{d^{i_1}\hat{y}_2}{dt^{i_2}} = \hat{C}_2\tilde{A}^{i_1}x + \hat{C}_2\tilde{A}^{i_1-1}\tilde{B}_1v_1$$

where $\tilde{A} = A + BF$, $\tilde{B} = BG$, $[\tilde{B}_1 \mid \tilde{B}_2] = \tilde{B}$, and $y' = [y_1', \hat{y}_2']$. Now, abuse notation and re-name $A = \tilde{A}$, $B = \tilde{B}$, $u = v$.
$u = v$.

R_k is invariant under regular feedback. Hence, from (4.1) and (A.1), one deduces that $\hat{C}_2A^jB_2 = 0$ all $0 \le j \le i_2 - 2$. Therefore,

$$\frac{d^{i_2}\hat{y}_2}{dt^{i_2}} = \hat{C}_2A^{i_2}x + \hat{C}_2A^{i_2-1}B_2u_2 + \sum_{j=0}^{i_2-i_1} \hat{C}_2A^{(i_2-1-j)}\frac{d^ju_1}{dt^j}$$

where rank $\hat{C}_2A^{i_2-1}B_2 = r_{i_2}$. Once again, re-order the rows of $\hat{y}_2$ so that the first r_{i_2} rows of $\hat{C}_2A^{i_2-1}B_2$ are linearly independent. Write $\hat{C}_2' = [C_2', \hat{C}_3']$ where C_2 has r_{i_2} rows. Let $u = F_2x + G_2v$ be any regular static feedback such that, upon writing $v' = [v_1', v_2', v_3']$, v_1 having r_{i_1} components, and v_2 having r_{i_2} components, one has

$$v_1 = u_1$$

$$v_2 = C_2A^{i_2}x + C_2A^{i_2-1}B_2u_2;$$

this is always possible because rank $C_2A^{i_2-1}B_2 = r_{i_2}$.

Proceeding in such a manner, the right-invertibility of Σ leads to (4.5). The required feedback is simply the composition of the feedbacks calculated at each step. □

APPENDIX B: PROOF OF LEMMA 2.1(d)

The necessity being obvious, only the sufficiency will be proved. Consider the i-th subsystem

$$\Sigma^i: \begin{array}{l} \dot{x} = Ax + Bu \\ y^i = C^ix \end{array},$$

which has p_i zeros at infinity. Discard as many components of the output as necessary to obtain a system which still has p_1^i zeros at infinity, but which is now right-invertible. Denote this subsystem by

$$\hat{\Sigma}^i: \begin{array}{l} \dot{x} = Ax + Bu \\ \hat{y}^i = \hat{C}^ix \end{array},$$

and define $\hat{y} = (\hat{y}^{1\prime}, \ldots, \hat{y}^{v\prime})'$. This gives rise to the system

$$\hat{\Sigma}: \begin{array}{l} \dot{x} = Ax + Bu \\ \hat{y} = \hat{C}x \end{array}. \tag{B.1}$$

Either using transfer function arguments, or the differential algebraic approach of Fliess (1985, 1986), one can show that a compensator input–output decouples (B.1) if and only if it decouples the original system (2.1). By construction, $\hat{\Sigma}$ has $\sum_{i=1}^{v} p_i^1$ output components, and p^1 zeros at infinity. Hence, the relationship $\sum_{i=1}^{v} p_i^1 = p^1$ implies that $\hat{\Sigma}$ is right-invertible, which in turn implies that $\hat{\Sigma}$ is dynamically input–output decouplable (Morse and Wonham, 1971).

Remark. A similar argument also works for nonlinear systems (Grizzle *et al.*, 1987).