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RANK INVARIANTS OF NONLINEAR SYSTEMS*

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Abstract. A linear algebraic framework for the analysis of rank properties of nonlinear systems is introduced. This framework gives a high-level interpretation of several existing algorithms built around the recursive computation of certain algebraic ranks associated with right-invertibility, left-invertibility, and dynamic decoupling. Furthermore, it can be used to establish links between these algorithms and the differential algebraic approach, as well as to solve some static and dynamic noninteracting control problems.

Key words. invertibility, decoupling, zeros at infinity, differential algebra, nonlinear systems analysis

AMS(MOS) subject classifications. primary 93C10; secondary 93B25

1. Introduction. Consider a nonlinear control system of the following form:

$$\Sigma: \begin{aligned} \dot{x} &= f(x)+g(x)u, &\quad (1.1a)\\ y &= h(x) &\quad (1.1b)\end{aligned}$$

where, for simplicity, $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^p$, and $f(\cdot)$, the columns of $g(\cdot)$ and the rows of $h(\cdot)$ are *meromorphic* functions of x (on all of $\mathbb{R}^n$); that is, they are elements of the fraction field $\mathcal{F}(x)$ of the ring of analytic functions of x. Our goal is to associate to such a system a chain of (nondifferential) vector spaces and show that it contains a rich amount of structural information about the system. More precisely, the subspaces will recover in a unified way, the inversion algorithm of Singh [1], the generic ranks of Nijmeijer [2], the dynamic decoupling algorithms of Descusse and Moog [3] and Nijmeijer and Respondek [4], and the differential output rank of Fliess [5], [6]. The approach adopted in the paper has been largely inspired by the differential vector spaces considered in [7] by Fliess.

To proceed, suppose that the input function $u(t)$ to the system (1.1) is N times continuously differentiable. Then by Taylor's Theorem,

$$u(t)=\sum_{k=0}^{N} u^{(k)}\frac{(t-t_0)^k}{k!}+R_N(t-t_0),$$

where t_0 is some initial point in time, $u^0 := u(t_0)$, $u^{(i+1)} := d/dt\, u^{(i)}(t)|_{t=t_0}$, and R_N is a remainder term. View $x, u, \cdots, u^{(n-1)}$ as variables and let $\mathcal{K}$ denote the field consisting of the set of rational functions of $(u, \cdots, u^{(n-1)})$ with coefficients that are meromorphic in x. Recall that given such a field, say in the variables $\nu=(\nu_1, \cdots, \nu_j)$, we define $\partial/\partial\nu_i$ acting on a meromorphic function $\eta(\nu)=\pi(\nu)/\theta(\nu)$, where $\pi(\cdot)$ and $\theta(\cdot)$ are analytic, by the usual quotient rule of calculus,

$$\frac{\partial}{\partial\nu_i}\frac{\pi(\nu)}{\theta(\nu)} := \frac{\left(\theta(\nu)\dfrac{\partial}{\partial\nu_i}\pi(\nu)-\pi(\nu)\dfrac{\partial}{\partial\nu_i}\theta(\nu)\right)}{\theta^2(\nu)}. \quad (1.2)$$

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Then we define the differential of η by

$$d\eta(\nu):=\sum_{i=1}^{j}\frac{\partial\eta(\nu)}{\partial\nu_i}\,d\nu_i. \tag{1.3}$$

Returning to the system (1.1), we define in a natural way,

$$\dot y=\dot y(x,u)=\frac{\partial h}{\partial x}[f(x)+g(x)u], \tag{1.4a}$$

$$\begin{aligned} y^{(k+1)}&=y^{(k+1)}(x,u,\cdots,u^{(k)})\\ &=\frac{\partial y^{(k)}}{\partial x}[f(x)+g(x)u]+\sum_{i=0}^{k-1}\frac{\partial y^{(k)}}{\partial u^{(i)}}u^{(i+1)}. \end{aligned} \tag{1.4b}$$

Note that $\dot y,\cdots,y^{(n)}$ so defined have components in the field $\mathscr{K}$.

Let $\mathscr{E}$ denote the vector space (over $\mathscr{K}$) spanned by $\{dx, du,\cdots, du^{(n-1)}\}$. It is essential to remark that this is an ordinary, nondifferential vector space as opposed to the setting proposed in [7]. Now we introduce the chain of subspaces $\mathscr{E}_0\subset\cdots\subset\mathscr{E}_n$ of $\mathscr{E}$ by

$$\mathscr{E}_k:=\text{span}\,\{dx, d\dot y,\cdots, dy^{(k)}\} \tag{1.5}$$

and define the associated list of dimensions $\rho_0\leqq\cdots\leqq\rho_n$ by

$$\rho_k:=\dim\mathscr{E}_k. \tag{1.6}$$

It is important to note that in (1.5) and (1.6) the span and dimension are both taken with respect to the field $\mathscr{K}$, and *not* the real numbers. Hence ρ_k is a well-defined integer and is not a function of $x, u,\cdots, u^{(n-1)}$. Note also that we abuse notation slightly because $\mathscr{E}_0:=\text{span}\,\{dx, dy\}$, which is easily seen to be equal to span $\{dx\}$ since the output y only depends on x. Finally, in the above, as well as in all that follows, "d" of a vector or vector valued function means "d" of each of its components; that is, $\mathscr{E}_0=\text{span}\,\{dx_1,\cdots, dx_n\}$, etc.

In the sequel we argue that the chain of subspaces $\mathscr{E}_0\subset\cdots\subset\mathscr{E}_n$ gives a linear algebraic framework that clarifies many structural properties of nonlinear systems and leads to a synthesis of many previous works on rank invariants of nonlinear systems [1]-[9].

2. Four concepts becomes one. For a linear system, there are many equivalent approaches to defining (or characterizing) its rank. More or less, all possible approaches have been extended to nonlinear systems in an effort to understand such properties as right-invertibility, left-invertibility, dynamic decoupling, etc. In general, these extensions lead to distinct notions when broadened to the class of nonlinear systems [1], [10]. However, we will show that the linear algebraic framework of § 1 can be used to establish the equivalence of four of them. A first attempt at this, using cruder tools, was made in [11].

2.1. Jacobian matrices. In [2], Nijmeijer considers systems of the form (1.1) where $f(\cdot)$, the columns of $g(\cdot)$ and the rows of $h(\cdot)$ are *analytic* functions of x. He defines $\dot y,\cdots, y^{(n)}$ as before, and introduces

$$J_k(x,u,\cdots,u^{(k-1)}):=\frac{\partial(\dot y,\cdots,y^{(k)})}{\partial(u,\cdots,u^{(k-1)})}. \tag{2.1}$$

J_k is an analytic function of its arguments; hence, we can define

$$R_k:=\text{generic rank (over the real numbers) of } J_k(x,u,\cdots,u^{(k-1)}). \tag{2.2}$$

Nijmeijer shows that $R_{k+1} - R_k$ is a nondecreasing sequence of integers and says that *(1.1)* is right-invertible if $R_n - R_{n-1} = p$, the number of scalar output components. He goes on to relate the integers R_k, for a restricted class of systems, to a set of integer invariants coming from the Δ^*-algorithm; that set, in analogy with results from linear system theory, was termed the "structure at infinity."

We now relate the list of generic ranks $\{R_k\}$ to the list of integers $\{\rho_k\}$ defined in (1.6). The first step is to observe that R_k is equal to the dimension of the row span of J_k, where the field is taken as $\mathcal{K}$. That is, if we define

$$\mathcal{V}_k := \operatorname{span}\left\{\frac{\partial \dot{y}}{\partial u}\, du, \cdots, \sum_{i=0}^{k-1} \frac{\partial y^{(k)}}{\partial u^{(i)}}\, du^{(i)}\right\}, \tag{2.3}$$

then $R_k = \dim \mathcal{V}_k$. It follows immediately that for $k \geqq 1$

$$\mathcal{E}_k = \operatorname{span}\{dx\} \oplus \mathcal{V}_k\,; \tag{2.4}$$

hence, we have the following theorem.

THEOREM 2.1. $\rho_k = n + R_k$, $1 \leqq k \leqq n$.

This yields the following corollary.

COROLLARY 2.2. *System* (1.1) *is right invertible in the sense of Nijmeijer if and only if* $\dim \mathcal{E}_n - \dim \mathcal{E}_{n-1} = p$, *the number of scalar output components.*

In closing this subsection, we note that the subspaces $\mathcal{V}_k$ give a linear algebraic interpretation of the Jacobian matrices J_k. In the case considered in this section where f, g, and h are analytic, both $\mathcal{E}_k$ and $\mathcal{V}_k$ can be viewed as analytic codistributions on the manifold $M \times T^{(n-1)}U$, where $T^{(n-1)}U$ is the $(n-1)$th order tangent bundle [12, Chap. 1] of the input manifold $U = \mathbb{R}^m$. However, it is easy to see that $\mathcal{E}_k$ is always involutive, whereas $\mathcal{V}_k$, the projection of $\mathcal{E}_k$ along span $\{dx\}$, does not in general enjoy this property. This gives us some reason to believe that the $\mathcal{E}_k$'s are more intrinsic, and certainly indicates that they are more amenable to analysis.

2.2. The inversion algorithm. In [1], Singh introduces an algorithm for calculating the left-inverse of a nonlinear system; his algorithm is in fact a generalization of previous algorithms, due to Silverman [13] and Hirschorn [14], that are only applicable under some restrictive conditions. This algorithm has since been taken up by different authors and has been used to define a finite zero structure for nonlinear systems [15] (important for certain stabilization problems), and a structure at infinity [9] (important for noninteracting control problems and model matching).

In the following, it will be shown that the inversion algorithm actually constructs, step by step, a *basis* for the chain of subspaces $\mathcal{E}_0 \subset \cdots \subset \mathcal{E}_n$. This shows that the chain $\mathcal{E}_0 \subset \cdots \subset \mathcal{E}_n$ contains all of the above-cited structural information yielded by the inversion algorithm, and also confirms the earlier claim that the chain $\mathcal{E}_0 \subset \cdots \subset \mathcal{E}_n$ embodies important structural information on a high level, independently of any particular algorithmic considerations.

The inversion algorithm as detailed in [15] is now given, with the exception that, instead of allowing a large class of analytic transformations, we will use a particular meromorphic transformation. This idea was first sketched in [9].

Step 1. Calculate

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] \tag{2.5}$$

and write it as

$$\dot{y} =: a_1(x) + b_1(x)u. \tag{2.6}$$

Define

$$s_1 := \operatorname{rank} b_1(x), \tag{2.7}$$

where the rank is taken over the field of meromorphic functions of x. Permute, if necessary, the components of the output so that the first s_1 rows of $b_1(x)$ are linearly independent. Decompose y as

$$\dot{y} = \begin{pmatrix} \dot{\tilde{y}}_1 \\ \dot{\hat{y}}_1 \end{pmatrix}, \tag{2.8}$$

where $\dot{\tilde{y}}_1$ consists of the first s_1 rows of $\dot{y}$. Since the last rows of $b_1(x)$ are linearly dependent upon the first s_1 rows, we can write

$$\dot{\tilde{y}}_1 = \tilde{a}_1(x) + \tilde{b}_1(x)u, \tag{2.9a}$$

$$\dot{\hat{y}}_1 = \dot{\hat{y}}_1(x, \dot{\tilde{y}}_1), \tag{2.9b}$$

where the last equation is affine in $\dot{\tilde{y}}_1$. Finally, set $\tilde{B}_1(x) := \tilde{b}_1(x)$.

Step $k+1$. Suppose that in Steps 1 through k, $\dot{\tilde{y}}_1, \cdots, \tilde{y}_k^{(k)}, \hat{y}_k^{(k)}$ have been defined so that

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u, \\ &\vdots \\ \tilde{y}_k^{(k)} &= \tilde{a}_k(x, \{\tilde{y}_i^{(j)} \mid 1 \leqq i \leqq k-1, i \leqq j \leqq k\}) \\ &\quad + \tilde{b}_k(x, \{\tilde{y}_i^{(j)} \mid 1 \leqq i \leqq k-1, i \leqq j \leqq k-1\})u, \\ \hat{y}_k^{(k)} &= \hat{y}_k^{(k)}(x, \{\tilde{y}_i^{(j)} \mid 1 \leqq i \leqq k, i \leqq j \leqq k\}) \end{aligned} \tag{2.10}$$

and so that they are rational functions of $\tilde{y}_i^{(j)}$ with coefficients in the field of meromorphic functions of x. Suppose also that the matrix $\tilde{B}_k := [\tilde{b}_1^T, \cdots, \tilde{b}_k^T]^T$ has full rank equal to s_k. Then calculate

$$\hat{y}_k^{(k+1)} = \frac{\partial}{\partial x} \hat{y}_k^{(k)}[f(x) + g(x)u] + \sum_{i=1}^{k} \sum_{j=i}^{k} \frac{\partial \hat{y}_k^{(k)}}{\partial \tilde{y}_i^{(j)}} \tilde{y}_i^{(j+1)} \tag{2.11}$$

and write it as

$$\hat{y}_k^{(k+1)} = a_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \leqq i \leqq k, i \leqq j \leqq k+1\}) + b_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \leqq i \leqq k, i \leqq j \leqq k\})u. \tag{2.12}$$

Define $B_{k+1} := [\tilde{B}_k^T, b_{k+1}^T]^T$, and

$$s_{k+1} := \operatorname{rank} B_{k+1}, \tag{2.13}$$

where the rank is taken with respect to the field of rational functions of $\{\tilde{y}_i^{(j)} \mid 1 \leqq i \leqq k-1, i \leqq j \leqq k\}$ with coefficients in the field of meromorphic functions of x. Permute, if necessary, the components of $\hat{y}_k^{(k+1)}$ so that the first s_{k+1} rows of B_{k+1} are linearly independent. Decompose $\hat{y}_k^{(k+1)}$ as

$$\hat{y}_k^{(k+1)} = \begin{pmatrix} \tilde{y}_{k+1}^{(k+1)} \\ \hat{y}_{k+1}^{(k+1)} \end{pmatrix}, \tag{2.14}$$

where $\tilde{y}_{k+1}^{(k+1)}$ consists of the first $(s_{k+1} - s_k)$ rows. Since the last rows of $B_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \leqq i \leqq k, i \leqq j \leqq k\})$ are linearly dependent on the first s_{k+1} rows, we can write

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u, \\ &\vdots \\ \tilde{y}_{k+1}^{(k+1)} &= \tilde{a}_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \leqq i \leqq k, i \leqq j \leqq k+1\}) \\ &\quad + \tilde{b}_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \leqq i \leqq k, i \leqq j \leqq k\})u, \\ \hat{y}_{k+1}^{(k+1)} &= \hat{y}_{k+1}^{(k+1)}(x, \{\tilde{y}_i^{(j)} \mid 1 \leqq i \leqq k+1, i \leqq j \leqq k+1\}), \end{aligned} \tag{2.15}$$

where once again everything is rational in $\tilde{y}_i^{(j)}$. Finally, set $\tilde{B}_{k+1} := [\tilde{B}_k^T, \tilde{b}_{k+1}^T]^T$.

End of Step $k+1$.

It is now possible to state and prove the main result of this subsection.

THEOREM 2.3. *For each* $1 \leqq k \leqq n$:

(a) $\{dx, \{d\tilde{y}_i^{(j)} \mid 1 \leqq i \leqq k,\ i \leqq j \leqq k\}\}$ *is a basis for* $\mathscr{E}_k$;

(b) $\dim \mathscr{E}_k = n + s_1 + \cdots + s_k$.

Proof. Part (b) is an immediate consequence of (a), which will be proved by induction. For $k=1$, the statement is obvious. Suppose that (a) holds at Step k; it will now be shown that it also holds at Step $k+1$. By construction, $\tilde{B}_{k+1}$ is a rational function of $\{\tilde{y}_i^{(j)} \mid 1 \leqq i \leqq k,\ i \leqq j \leqq k\}$ with coefficients in the field of meromorphic functions of x. As $\tilde{y}_i^{(j)}$ is a rational function of $(u, \cdots, u^{(n-1)})$ with coefficients in the field of meromorphic functions of x, it therefore follows that $\tilde{B}_{k+1}$ is also. Since from Step k, $\{dx, \{d\tilde{y}_i^{(j)} \mid 1 \leqq i \leqq k,\ i \leqq j \leqq k\}\}$ is a linearly independent set over $\mathscr{K}$, it follows easily that $\tilde{B}_{k+1}$, when viewed as a rational function of $(u, \cdots, u^{(n-1)})$, has rank s_{k+1} over the field $\mathscr{K}$. Using (2.15), we show readily that

$$\begin{aligned} &\dim \operatorname{span} \{dx, \{d\tilde{y}_i^{(j)} \mid 1 \leqq i \leqq k,\ i \leqq j \leqq k\}\} \\ &\qquad = \dim \operatorname{span} \{dx, \{\tilde{B}_\ell\, du^{(k+1-\ell)} \mid 1 \leqq \ell \leqq k+1\}\}. \end{aligned} \tag{2.16}$$

The dimension on the right-hand side of (2.16) is easily seen to be $n + s_1 + \cdots + s_{k+1}$, showing that, indeed, the vectors on the left-hand side of (2.16) are linearly independent since there are precisely $n + s_1 + \cdots + s_{k+1}$ elements; it only remains to show that they span $\mathscr{E}_{k+1}$. By its definition,

$$\mathscr{E}_{k+1} = \mathscr{E}_k + \operatorname{span} \{dy^{(k+1)}\}. \tag{2.17}$$

Using (2.15) once again, we can write this as follows:

$$\mathscr{E}_{k+1} = \mathscr{E}_k + \operatorname{span} \{d\tilde{y}_i^{(k+1)} \mid 1 \leqq i \leqq k+1\}, \tag{2.18}$$

which, coupled with the induction hypothesis, completes the proof. □

COROLLARY 2.4. *System* (1.1) *is left-invertible in the sense of Singh if and only if* $\dim \mathscr{E}_n - \dim \mathscr{E}_{n-1} = m$, *the number of scalar input components.*

2.3. The dynamic extension algorithm. The relationship between dynamic input-output decoupling and right invertibility has been clarified recently by Fliess [5], whose work has inspired several authors to develop concrete algorithms for the explicit construction of a dynamic compensator yielding a noninteractive system [3], [4]. Here we will give a simplified version of these algorithms, separating clearly their basic operations of differentiating the outputs, performing static-state feedback, and adding integrators on selected components of the input. This simplified version will still yield a decoupling compensator whenever one exists, but does introduce more integrators than the algorithms previously cited. We will show that it explicitly produces a basis for the chain of subspaces $\mathscr{E}_0 \subset \cdots \subset \mathscr{E}_n$. This shows yet another way in which the chain $\mathscr{E}_0 \subset \cdots \subset \mathscr{E}_n$ incorporates important structural information on a high level—in this case, that information pertraining to dynamic decoupling.

The dynamic extension algorithm is now presented.

Step 1. Let Σ_0 denote the system (1.1). Calculate

$$\dot{y} = \frac{\partial h(x)}{\partial x}[f(x) + g(x)u] \tag{2.19}$$

and write it as

$$\dot{y} = a_1(x) + b_1(x)u. \tag{2.20}$$

Define

(2.21) $$\sigma_1 = \text{rank } b_1(x),$$

where the rank is taken over the field of meromorphic functions of x. Permute, if necessary, the components of the output so that the first σ_1 rows of $b_1(x)$ are linearly independent. Decompose $\dot{y}$ as

(2.22) $$\dot{y} = \begin{pmatrix} \dot{\bar{y}}_1 \\ \dot{\hat{y}}_1 \end{pmatrix} = \begin{pmatrix} \bar{a}_1(x) + \bar{b}_1(x)u \\ \hat{a}_1(x) + \hat{b}_1(x)u \end{pmatrix}.$$

Let

(2.23) $$u = \alpha_1(x) + \beta_1(x)v_1$$

be any static state feedback such that

(i) $\beta_1(x)$ is invertible over the field of meromorphic functions in x;

(ii) $\dot{\bar{y}}_1 = \bar{v}_1$.

Such a feedback always exists.

For the resulting closed-loop system, we can write

(2.24) $$\dot{\hat{y}}_1 = \dot{\hat{y}}_1(x, \bar{v}_1),$$

since otherwise the rank of $\partial\dot{y}/\partial v_1$ would exceed σ_1. Moreover, $\dot{\hat{y}}_1$ is affine in $\bar{v}_1$. Now introduce a dynamic extension by

(2.25a) $$\dot{\bar{v}}_1 = \bar{u}_1$$

and rename the remaining components of v_1:

(2.25b) $$\hat{v}_1 = \hat{u}_1.$$

Finally, let Σ_1 denote the system consisting of Σ_0, the static-state feedback (2.23), and the dynamic extension (2.25). Its state is given by $x_1 = \binom{x}{\bar{v}_1}$, its input is $u_1 = \binom{\bar{u}_1}{\hat{u}_1}$, and the output remains $y = h(x)$. We will denote it as

(2.26) $$\Sigma_1: \begin{array}{l} \dot{x}_1 = f_1(x_1) + g_1(x_1)u_1, \\ y = h(x). \end{array}$$

Σ_1 is a dynamic extension of Σ_0 and has the property that $\dot{y} = \dot{y}(x_1)$ is affine in $\bar{v}_1$.

Step $k+1$. Suppose that in Step k the system Σ_k has been constructed such that

(2.27) $$\Sigma_k: \begin{array}{l} \dot{x}_k = f_k(x_k) + g_k(x_k)u_k, \\ y = h(x) \end{array}$$

and $y_k^{(k)} = y_k^{(k)}(x_k)$ is a rational function of $\bar{v}_1, \cdots, \bar{v}_k$ with coefficients in the field of meromorphic functions of x. Then, calculate

(2.28) $$y^{(k+1)} = \frac{\partial y^{(k)}}{\partial x_k}[f_k(x_k) + g_k(x_k)u_k]$$

and rewrite (2.28) as

(2.29) $$y^{(k+1)} = a_{k+1}(x_k) + b_{k+1}(x_k)u_k.$$

Define

(2.30) $$\sigma_{k+1} = \text{rank } b_{k+1}(x_k),$$

where the rank is over the field of rational functions of $\bar{v}_1, \cdots, \bar{v}_k$ with coefficients in the field of meromorphic functions of x. Permute, if necessary, the last $p-\sigma_k$ components of the output so that the first σ_{k+1} rows of $b_{k+1}(x_k)$ are linearly independent. Decompose $y^{(k+1)}$ as

$$y^{(k+1)} = \begin{pmatrix} \bar{y}_{k+1}^{(k+1)} \\ \hat{y}_{k+1}^{(k+1)} \end{pmatrix} = \begin{pmatrix} \bar{a}_{k+1}(x_k) + \bar{b}_{k+1}(x_k)u_k \\ \hat{a}_{k+1}(x_k) + \hat{b}_{k+1}(x_k)u_k \end{pmatrix}. \tag{2.31}$$

Let

$$u_k = \alpha_{k+1}(x_k) + \beta_{k+1}(x_k)v_{k+1} \tag{2.32}$$

be any static-state feedback such that

(i) $\beta_{k+1}(x_k)$ is invertible over the field of rational functions of $\bar{v}_1, \cdots, \bar{v}_k$ with coefficients in the field of meromorphic functions of x.

(ii) $\bar{y}_{k+1}^{(k+1)} = \bar{v}_{k+1}$.

Such a feedback always exists.

For the resulting closed-loop system, we can always write

$$\hat{y}_{k+1}^{(k+1)} = \hat{y}_{k+1}^{(k+1)}(x_k, \bar{v}_{k+1}), \tag{2.33}$$

since otherwise the rank of $\partial y^{(k+1)}/\partial v_{k+1}$ would exceed σ_{k+1}. Moreover, $y^{(k+1)}$ is a rational function of $\bar{v}_1, \cdots, \bar{v}_{k+1}$. Now introduce a dynamic extension by

$$\dot{\bar{v}}_{k+1} = \bar{u}_{k+1} \tag{2.34a}$$

and rename the remaining components of v_{k+1}:

$$\hat{v}_{k+1} = \hat{u}_{k+1}. \tag{2.34b}$$

Finally, let Σ_{k+1} denote the system consisting of Σ_k, the static state feedback (2.32) and the dynamic extension (2.34). Its state is given by $x_{k+1} = \binom{x_k}{\bar{v}_{k+1}}$, its input is $u_{k+1} = \binom{\bar{u}_{k+1}}{\hat{u}_{k+1}}$, and the output remains $y = h(x)$. Σ_{k+1} is then a dynamic extension of Σ_k, has the property that $y^{(k+1)} = y^{(k+1)}(x_{k+1})$, and is a rational function of $\bar{v}_1, \cdots, \bar{v}_{k+1}$.

End of Step $k+1$.

We now have the following theorem.

THEOREM 2.5. *For each* $1 \leqq k \leqq n$:

(a) $\{dx, d\dot{\bar{y}}_1, \cdots, d\bar{y}_k^{(k)}\}$ *is a basis for* $\mathscr{E}_k$.

(b) $\dim \mathscr{E}_k = n + \sigma_1 + \cdots + \sigma_k$.

Proof. Part (b) is an immediate consequence of (a). From (2.33), one has that

$$\mathscr{E}_k = \operatorname{span}\{dx, d\dot{\bar{y}}_1, \cdots, d\bar{y}_k^{(k)}\}, \tag{2.35}$$

which by definition of $\bar{v}_j$ gives

$$\mathscr{E}_k = \operatorname{span}\{dx, d\bar{v}_1, d\bar{v}_2, \cdots, d\bar{v}_k\}. \tag{2.36}$$

Hence, it suffices to show that $\{dx, d\bar{v}_1, \cdots, d\bar{v}_k\}$ is a linearly independent set for each $1 \leqq k \leqq n$. This follows immediately from Lemma 2.6.

LEMMA 2.6. *For each* $1 \leqq k \leqq n-1$,

$$\begin{aligned} &\operatorname{span}\{dx, du, \cdots, du^{(n-1)}\} \\ &\qquad = \operatorname{span}\{dx, d\bar{v}_1, \cdots, d\bar{v}_k, du_k, \cdots, du_k^{(n-1-k)}, d\hat{v}_k^{(n-k)}, \cdots, d\hat{v}_1^{(n-1)}\}, \end{aligned} \tag{2.37}$$

where all spans are with respect to the field $\mathscr{K}$.

Proof. Equations (2.23) and (2.25) yield

$$\begin{aligned}\operatorname{span}\{dx, du, \cdots, du^{(n-1)}\}\\ &= \operatorname{span}\{dx, d\bar{v}_1, \{d\hat{v}_1, d\dot{\bar{v}}_1\}, \cdots, \{d\hat{v}_1^{(n-2)}, d\bar{v}_1^{(n-1)}\}, d\hat{v}_1^{(n-1)}\}\\ &= \operatorname{span}\{dx, d\bar{v}_1, du_1, \cdots, du_1^{(n-2)}, d\hat{v}_1^{(n-1)}\},\end{aligned}$$

which establishes (2.37) for $k=1$.

Now suppose that (2.37) holds at k. Then, in particular, $\{dx, d\bar{v}_1, \cdots, d\bar{v}_k\}$ is a linearly independent set. After we note that each component of $x_k = (x^T, \bar{v}_1^T, \cdots, \bar{v}_k^T)^T$ is a rational function of $(u, \cdots, u^{(n-1)})$, this gives us that b_{k+1} has full rank over $\mathcal{K}$. Hence, using (2.32) and (2.34), we have

$$\begin{aligned}\operatorname{span}\{d\bar{x}_k, du_k, \cdots, du_k^{(n-1-k)}\}\\ &= \operatorname{span}\{d\bar{x}_k, d\bar{v}_{k+1}, \{d\hat{v}_{k+1}, d\dot{\bar{v}}_{k+1}\}, \cdots, \{d\hat{v}_{k+1}^{(n-k-2)}, d\bar{v}_{k+1}^{(n-1-k)}\}, d\hat{v}_{k+1}^{(n-1-k)}\}\\ &= \operatorname{span}\{d\bar{x}_k, d\bar{v}_{k+1}, du_{k+1}, \cdots, du_{k+1}^{(n-k-2)}, d\hat{v}_{k+1}^{(n-1-k)}\},\end{aligned}$$

which establishes (2.37) at $k+1$. □

2.4. Convergence of the chain $\mathscr{E}_0 \subset \cdots \subset \mathscr{E}_n$. In a completely analogous fashion to the way the chain $\mathscr{E}_0 \subset \cdots \subset \mathscr{E}_n$ is defined in (1.5), we could define an extended chain $\mathscr{E}_0 \subset \cdots \subset \mathscr{E}_n \subset \cdots \subset \mathscr{E}_{n+k}$, for any finite integer $k \geqq 1$. The underlying field $\mathcal{K}_{n+k}$ would then consist of the set of rational functions of $(u, \cdots, u^{(n+k-1)})$ with coefficients that are meromorphic in x. We will show, however, that no new structural information is obtained by extending the original chain; hence, we are justified in terminating the chain at n, the dimension of the state space of (1.1).

THEOREM 2.7. *For all finite integers $k \geqq 1$:*

(a) $\{dx, d\dot{\bar{y}}_1, \cdots, d\bar{y}_n^{(n)}, d\bar{y}_n^{(n+1)}, \cdots, d\bar{y}_n^{(n+\ell)}\}$ *is a basis for* $\mathscr{E}_{n+\ell}$, $1 \leqq \ell \leqq k$.

(b) $\dim \mathscr{E}_{n+\ell} = \dim \mathscr{E}_n + \ell \cdot \rho^*$, $1 \leqq \ell \leqq k$, *where* $\rho^* = \dim \mathscr{E}_n - \dim \mathscr{E}_{n-1}$.

The proof will be broken down into two lemmas. The functions $(\bar{y}_i^T, \hat{y}_i^T) = y^T$ continue to denote the blocks of the output constructed in the ith step of the dynamic extension algorithm; $\bar{y}_i^{(j)}$, $\hat{y}_i^{(j)}$ denote their jth-order derivatives along the dynamics of the system.

LEMMA 2.8. *There exists an integer $1 \leqq N \leqq n$ such that*

$$d\hat{y}_n^{(N)} \in \operatorname{span}_{\mathcal{K}}\{d\hat{y}_n, \cdots, d\hat{y}_n^{(N-1)}, d\dot{\bar{y}}_1, \cdots, d\bar{y}_n^{(N)}\}. \tag{2.38}$$

Proof. From the dynamic extension algorithm, we have that $\hat{y}_k^{(k)} = \hat{y}_k^{(k)}(x, \dot{\bar{y}}_1, \cdots, \bar{y}_k^{(k)})$. Hence, (2.38) holds if

$$\frac{\partial \hat{y}_n^{(N)}}{\partial x} dx \in \operatorname{span}_{\mathcal{K}}\left\{\frac{\partial \hat{y}_n}{\partial x} dx, \cdots, \frac{\partial \hat{y}_n^{(N-1)}}{\partial x} dx\right\}. \tag{2.39}$$

Since the right-hand side of (2.39) can be at most n-dimensional, there must exist $1 \leqq N \leqq n$ such that (2.39) holds. Hence, the result is proved. □

For each integer j, $N \leqq j \leqq n+k$, define the field $\mathcal{K}_j$ to be the set of rational functions of $(u, \cdots, u^{(j-1)})$ with coefficients that are meromorphic in x.

LEMMA 2.9. *With N as in Lemma 2.8 and for all integers j, $N \leqq j \leqq n+k$,*

$$d\hat{y}_n^{(j)} \in \operatorname{span}_{\mathcal{K}_j}\{d\hat{y}_n, \cdots, d\hat{y}_n^{(N-1)}, d\dot{\bar{y}}_1, \cdots, d\bar{y}_j^{(j)}\}, \tag{2.40}$$

where, for $j \geqq n$, $\bar{y}_j = \bar{y}_n$.

Proof. Since all the functions appearing in (2.38) are rational functions of $(u, \cdots, u^{(N-1)})$, (2.38) also holds with $\mathcal{K}$ replaced by $\mathcal{K}_N$. Hence there exist coefficients $\alpha_i, \beta_i \in \mathcal{K}_N$ such that

$$d\hat{y}_n^{(N)} = \sum_{i=0}^{N-1} \alpha_i \, d\hat{y}_n^{(i)} + \sum_{i=1}^{N} \beta_i \, d\bar{y}_i^{(i)}. \tag{2.41}$$

Hence,

$$d\hat{y}_n^{(N+1)} = \sum_{i=0}^{N-1} (\dot{\alpha}_i \, d\hat{y}_n^{(i)} + \alpha_i \, d\hat{y}_n^{(i+1)}) + \sum_{i=1}^{N} (\dot{\beta}_i \, d\bar{y}_i^{(i)} + \beta_i \, d\bar{y}_i^{(i+1)}). \tag{2.42}$$

Combining (2.41) with the fact that $\dot{\alpha}_i$, $\dot{\beta}_i$ are in $\mathcal{K}_{N+1}$, we easily see that (2.42) establishes (2.40) for $j = N+1$. In a similar manner we complete the proof of Lemma 2.9. □

To complete the proof of Theorem 2.7, note that Lemma 2.9 establishes that $\mathcal{E}_{n+\ell}$, $1 \leqq \ell \leqq k$ is indeed spanned by the vectors given in (a). We prove that they are independent using the same reasoning employed in the proof of Theorem 2.3. Part (b) follows from (a) by counting the number of basis elements.

Remark 2.10. If we introduce the finite chain of subspaces $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_n$ of $\mathcal{E}$ by $\mathcal{F}_k = \text{span}\{dy, d\dot{y}, \cdots, dy^{(k)}\}$, then we can show that $\rho^* = \dim \mathcal{F}_n - \dim \mathcal{F}_{n-1}$. However, the inversion algorithm and the dynamic extension algorithm do not calculate bases for this chain.

2.5. Differential output rank. In 1985, Fliess introduced a new approach, centered around differential algebra, to the analysis of nonlinear systems, [5], [16], the proper formalism being perhaps field theoretic [17]. He was the first to define clearly and precisely the fundamental notion of the *rank* of a nonlinear system; he accomplished this by considering the output components to be dependent if they satisfied a nontrivial (nonlinear) differential equation. Fliess' notion of rank generalized to nonlinear systems the usual notion of the rank of a transfer matrix of a linear system, and played the same important role in leading to basic definitions of right invertibility and left invertibility, and important new results on dynamic feedback [5], [16]. All previous attempts in this regard lacked the power and elegance of the differential algebraic approach, being mainly based on algorithmic considerations.

Additional insights on the role of differential algebra in system theory have been contributed by Pommaret [18].

In this section, we will show that the integer $\rho^* = \dim \mathcal{E}_n - \dim \mathcal{E}_{n-1}$ of Theorem 2.7 is actually the differential output rank for those systems that meet the requirements both of § 1 and of the differential algebraic approach. More precisely, we suppose that the system (1.1) satisfies the following assumptions:

(A1) f, g and h are meromorphic;

(A2) f, g and h are differentially algebraic (i.e., elementary transcendental) functions of their arguments [5], [19], [20];

(A3) The set $\mathbb{R}\langle y \rangle$ of all rational functions of $y_i^{(\ell)}$, $1 \leqq i \leqq p$, $\ell \geqq 0$, with coefficients in the field $\mathbb{R}$ is a differential field.

Here, the $y_i^{(\ell)}$ are defined as in (1.4). (It is apparently unknown whether (A1) implies (A3).)

To aid the reader, a few notions from differential algebra are briefly recalled. A finite set of elements $\zeta_1, \cdots, \zeta_k$ of $\mathbb{R}\langle y \rangle$ *is differentially algebraically dependent* if there exists a nonzero differential polynomial $\mathbf{P}$, with coefficients in $\mathbb{R}$, such that

$\mathbf{P}(\zeta_1, \cdots, \zeta_k) = 0$; that is, a nonzero polynomial in $\zeta_1, \cdots, \zeta_k$ and a finite number of their time derivatives. The *differential output rank*, denoted $d^0(\Sigma)$, is the maximum number of differentially algebraically independent elements of $\mathbb{R}\langle y\rangle$. This number is well defined [21].

Our main result relating the $d^0(\Sigma)$ to the chain $\mathscr{E}_0 \subset \cdots \subset \mathscr{E}_n$ will be an easy consequence of the following properties of the differential rank.

LEMMA 2.11 [5]-[7], [16]. (a) $d^0(\Sigma) \leqq \min\{m, p\}$.

(b) *If Σ' is obtained from Σ by applying an invertible static state feedback, whose components are differentially algebraic functions, then $d^0(\Sigma') = d^0(\Sigma)$.*

(c) *If Σ_e is obtained by adding a finite number k_i of integrators on each input channel i of Σ, then $d^0(\Sigma_e) = d^0(\Sigma)$.*

(d) *If for some $1 \leqq i \leqq m$, $\partial y^{(k)}/\partial u_i^{(\ell)} = 0$ for all $k \geqq 0$, $\ell \geqq 0$, then $d^0(\Sigma) \leqq \min\{m-1, p\}$.*

(e) *If for some finite integer k, $y^{(k)} = y^{(k)}(x, u)$, and rank $\partial y^{(k)}/\partial u = r$, then $d^0(\Sigma) \geqq r$.*

Proof. All of these points are essentially contained in [5]-[7], [16]. Part (a) follows from the definition of the differential output rank. Part (b) is true because $d^0(\Sigma') \leqq d^0(\Sigma)$ for any static-state feedback. Thus invertibility gives equality, since the inverse, being a well-defined (differentially algebraic) static-state feedback, yields $d^0(\Sigma) \leqq d^0(\Sigma')$. To prove (c), first note that

$$m = \text{diff. tr. } d^0 \frac{\mathbb{R}\langle u, y\rangle}{\mathbb{R}} = \text{diff. tr. } d^0 \frac{\mathbb{R}\langle u, y\rangle}{\mathbb{R}\langle y\rangle} + \text{diff. tr. } d^0 \frac{\mathbb{R}\langle y\rangle}{\mathbb{R}}, \tag{2.43}$$

which yields

$$d^0(\Sigma) = m - \text{diff. tr. } d^0 \frac{\mathbb{R}\langle u, y\rangle}{\mathbb{R}\langle y\rangle}. \tag{2.44}$$

Similarly,

$$d^0(\Sigma_e) = m - \text{diff. tr. } d^0 \frac{\mathbb{R}\langle v, y\rangle}{\mathbb{R}\langle y\rangle}. \tag{2.45}$$

This establishes the result because the right-hand sides of (2.44) and (2.45) are equal, since v constitutes a differential transcendence basis for $\mathbb{R}\langle u\rangle/\mathbb{R}$. Statement (d) follows from (a) because y is differentially algebraic over $\mathbb{R}\langle u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_m\rangle$. Finally, (e) implies that

$$\dim_{\mathscr{K}} \{d\dot{y}, \cdots, dy^{(k)}\} \geqq r, \tag{2.46}$$

which yields

$$\dim_{\mathbb{L}} \{d\dot{y}, \cdots, dy^{(k)}\} \geqq r, \tag{2.47}$$

where $\mathbb{L}$ is the field of rational functions in the indeterminates $\{y, \dot{y}, \cdots, y^{(k)}\}$ with coefficients in $\mathbb{R}$; note that $\mathbb{L} \subset \mathscr{K}$. Hence, by the results in [22], $d^0(\Sigma) \geq r$. □

THEOREM 2.12. *Suppose that system* (1.1) *satisfies Assumptions* (A1)-(A3). Then

$$d^0(\Sigma) = \dim \mathscr{E}_n - \dim \mathscr{E}_{n-1}. \tag{2.48}$$

Proof. Consider the extended system Σ_n constructed at the nth step of the dynamic extension algorithm. From (2.33), Σ_n has the property that rank $\partial y^{(n)}/\partial v_n = \sigma_n$. Hence, by (e) of Lemma 2.11, $d^0(\Sigma_n) \geqq \sigma_n$. By Theorem 2.7, $\sigma_{n+\ell} = \sigma_n$ for all $\ell \geqq 0$; therefore, y satisfies $\partial y^{(k)}/\partial \hat{v}_n^{(\ell)} = 0$ for all $k \geqq 0$, $\ell \geqq 0$. Thus, by (d) of Lemma 2.11, $d^0(\Sigma_n) \leqq \sigma_n$.

Hence, $d^0(\Sigma_n)=\sigma_n$. However, (b) and (c) of Lemma 2.11 then yield that $d^0(\Sigma)=d^0(\Sigma_n)=\sigma_n$, because Σ_n is constructed from Σ by successively applying invertible static feedbacks and finite strings of integrators. Finally, Theorem 2.5 yields $\sigma_n=\dim \mathscr{E}_n-\dim \mathscr{E}_{n-1}$. □

3. Structure at infinity and block decoupling. In this section, the framework of §§ 1 and 2 will be used to give an intrinsic "algebraic" definition of some important integer invariants associated with a nonlinear system, namely the so-called structure at infinity. An alternate approach to defining a structure at infinity has already been carried out in [8], using differential geometric techniques. When specialized to the class of linear systems, the geometric definition and the algebraic approach below both agree with the usual linear notion of the structure at infinity [23]. For general nonlinear systems, both are invariant under regular *static* state feedbacks. However, in contrast to the geometric approach, the algebraically based definition enjoys some additional properties that make it seem closer to the linear situation.

DEFINITION 3.1 [9]. The *number* σ_k *of zeros at infinity of order less than or equal to* k, $k\geqq 1$, is $\sigma_k=\dim \mathscr{E}_k-\dim \mathscr{E}_{k-1}$. The *structure at infinity* is given by the list $\{\sigma_1,\cdots,\sigma_n\}$.

Note that the total number of zeros at infinity σ_n corresponds precisely to the rank ρ^* of the system (1.1).

The notion of a *regular dynamic feedback* is introduced next. Note that when $q=0$ in the following definition, we recover the usual definition of a regular static feedback.

DEFINITION 3.2 [24]. The compensator

$$\dot z=F(x,z)+G(x,z)v,\qquad u=\alpha(x,z)+\beta(x,z)v, \tag{3.1}$$

where F, G, α, and β are meromorphic, $v\in\mathbb{R}^m$, and $z\in\mathbb{R}^q$ for a given integer q, is said to be *regular* if the composite system (1.1a) and (3.1), with v as the input and u as the output, has rank equal to m.

The structure at infinity of Definition 3.1 enjoys the following properties.

LEMMA 3.3. (a) *The rank* ρ^* *of the system* (1.1) *is equal to* σ_n, *the total number of zeros at infinity.*

(b) $\sigma_n\leqq\min\{m,p\}$, *the number of input and output components, respectively.*

(c) *The total number of zeros at infinity is invariant under regular dynamic feedback.*

Proof. Properties (a) and (b) are immediate from the results of § 2. To prove (c), first introduce the field $\mathscr{K}_e$ consisting of the set of rational functions of $(v,\cdots,v^{(n+q-1)})$ with coefficients which are meromorphic in x and z. Assume that for a given l, $0\leqq l\leqq n+q-1$, there exists i, $1\leqq i\leqq m$, such that

$$du_i^{(l)}\in\operatorname{span}_{\mathscr{K}_e}\{dx,du,\cdots,du^{(l-1)},du_j^{(l)},j\neq i\}.$$

Then, following the calculations involved in the proof of Lemma 2.9, we see that

$$du_i^{(k)}\in\operatorname{span}_{\mathscr{K}_e}\{dx,du,\cdots,du^{(k-1)},du_j^{(k)},j\neq i\}\quad\forall k\geqq l,$$

which contradicts the regularity assumption of the dynamic compensator. Hence, for every l, $\{dx,du,\cdots,du^{(l)}\}$ is a linearly independent set. Let $y_e=h(x)$ denote the output of the composite system Σ_e consisting of (1.1) and (3.1), and define

$$\mathscr{G}_k=\operatorname{span}_{\mathscr{K}_e}\{dx,dy_e,\cdots,dy_e^{(k)}\}.$$

By the chain rule,

$$\frac{\partial(x,y_e,\cdots,y_e^{(k)})}{\partial(x,z,v,\cdots,v^{(k-1)})}=\frac{\partial(x,y,\cdots,y^{(k)})}{\partial(x,u,\cdots,u^{(k-1)})}\,\frac{\partial(x,u,\cdots,u^{(k-1)})}{\partial(x,z,v,\cdots,v^{(k-1)})}. \tag{3.2}$$

Since $\{dx, du, \cdots, du^{(k-1)}\}$ are independent, (3.2) yields that

$$\dim \mathcal{G}_k = \operatorname{rank} \frac{\partial(x, y, \cdots, y^{(k)})}{\partial(x, u, \cdots, u^{(k-1)})},$$

so that

$$\dim \mathcal{G}_k = \dim \mathcal{E}_k. \tag{3.3}$$

Finally, let ρ_e^* denote the rank Σ_e. Then, following the reasoning employed in Lemma 2.8, we show that $\rho_e^* = \dim \mathcal{G}_{n+q} - \dim \mathcal{G}_{n+q-1}$, which yields the result in view of (3.3). □

Now consider a system whose outputs have been grouped into blocks:

$$\dot{x} = f(x) + g(x)u, \tag{1.1a}$$

$$y_i = h_i(x), \qquad 1 \leq i \leq \mu, \tag{3.4}$$

where $y_i \in \mathbb{R}^{p_i}$ and h_i is a meromorphic function of x. The system is said to be decoupled with respect to a given partition $u = (u_1^T, \cdots, u_\mu^T)^T$ of the input if u_i affects only y_i, $1 \leq i \leq \mu$; that is,

$$dy_i^{(k)} \in \operatorname{span}\{dx, du_i, \cdots, du_i^{(n-1)}\} \tag{3.5}$$

for $0 \leq k \leq n$. The decoupling problem is to find, if possible, a regular dynamic compensator and a partition of the new reference input such that the resulting closed-loop system is decoupled. If the compensator has dimension zero, the solution is said to be static; otherwise it is dynamic.

Using (3.5), we immediately obtain the following necessary condition for regular static block decoupling.

PROPOSITION 3.4 (see also [8]). *If the system* (1.1a), (3.4) *with block-partitioned outputs can be decoupled with a regular static state variable feedback, then*

$$\sigma_k = \sum_{i=1}^{\mu} \sigma_k^i, \tag{3.6}$$

where $\{\sigma_1^i, \cdots, \sigma_n^i\}$ *is the structure at infinity of the ith subsystem consisting of the dynamics* (1.1a) *and the output* y_i.

When the outputs y_i are scalar-valued, it is known that (3.6) is also sufficient (see, for example, [9], [25]). This condition is also known to be sufficient for general vector-valued outputs if we restrict our attention to the class of linear systems [26]. Indeed, (3.6) implies that we can perform the inversion algorithm for the overall system (1.1a), (3.4) by applying it to each of the individual subsystems y_i, for $1 \leq i \leq \mu$. Then, specializing equation (2.10) to the nth step of the algorithm and invoking linearity, we see that the $\tilde{b}_i$'s are constants, and the $\tilde{a}_i$'s are linear functions of x and various derivatives of y_i. Decoupling is accomplished by cancelling the dependence on x, and diagonalizing the matrix multiplying the inputs, via a static feedback.

The fact that condition (3.6) is not sufficient for general nonlinear systems is illustrated by the following example.

Example 3.5. Consider the system

$$\dot{x} = (u_1, x_4 + x_5 u_1, u_2, x_3 u_1, u_3)^T,$$

$$y_1 = (x_1, x_2)^T, \qquad y_2 = x_3.$$

We easily calculate that $\{\sigma_1, \sigma_2\} = \{2, 3\}$, and for the two subsystems $\{\sigma_1^1, \sigma_2^1\} = \{1, 2\}$ and $\{\sigma_1^2\} = \{1\}$. Thus, (3.6) is fulfilled. Nevertheless, a straightforward application of the results of Nijmeijer and Schumacher [8] shows that the system cannot be decoupled by any regular static-state feedback.

This example underlines the importance of the differential geometric approach in general, and the "geometric" structure at infinity in particular, for the study of static state feedback control problems, since the equivalent of (3.6) for the geometric structure at infinity constitutes a local necessary and sufficient condition (at regular points) [8]. On the other hand, algebraic methods seem to be better when we are studying dynamic feedback problems [3], [4], [27]. In this spirit, we have the following result that does not hold with the geometric version of the structure at infinity because it fails [10] the properties of Lemma 3.3.

THEOREM 3.6. *The system* (1.1a), (3.4) *can be decoupled with regular dynamic state feedback if and only if*

$$\rho = \sum_{i=1}^{\mu} \rho_i, \tag{3.7}$$

where ρ_i *denotes the rank of the subsystem* (1.1a) *with output* y_i.

Proof. Because the necessity is clear, only the sufficiency will be shown. For each block of outputs y_i, permute if necessary the components in such a way that $y_i = (\bar{y}_i^T, \hat{y}_i^T)^T$, dim $\bar{y}_i = \rho_i$, and on defining

$$\mathscr{E}_k^i = \text{span}\{dx, d\dot{y}_i, \cdots, dy_i^{(k)}\},$$

we have

$$\mathscr{E}_k^i = \text{span}\{dx, d\dot{\bar{y}}_i, \cdots, d\bar{y}_i^{(k)}\}$$

for all k, $1 \leqq k \leqq n$. This can always be accomplished with the help of the inversion algorithm of § 2.2. Then,

$$d\hat{y}_i^{(k)} \in \text{span}\{dx, d\dot{\bar{y}}_i, \cdots, d\bar{y}_i^{(k)}\}$$

and the rank of the subsystem (1.1a) with output $\bar{y}_i$ is equal to ρ_i. Let $\bar{\Sigma}$ denote the subsystem whose output is given by $(\bar{y}_1, \cdots, \bar{y}_\mu)$. We conclude that the rank of $\bar{\Sigma}$ is ρ and the original system (1.1a), (3.4) can be decoupled if and only if $\bar{\Sigma}$ can be. It follows from (3.7) that the rank of $\bar{\Sigma}$ equals the number of its scalar output components; thus, $\bar{\Sigma}$ can be row-wise decoupled [25].

4. Conclusions. In the recent literature, there have been many attempts to extend concepts and tools from the linear setting to the class of nonlinear systems. For example, Nijmeijer extends a definition of right invertibility based on the consideration of the sequence of Toeplitz matrices associated to a linear system. Singh extends the notion of left invertibility based on the input elimination idea of Silverman's algorithm. Descusse and Moog and Nijmeijer and Respondek use the idea of delaying the inputs via the addition of integrators, as does Wang [28], to achieve dynamic decoupling of nonlinear systems. Fliess uses differential algebra to extend the notion of the rank of a transfer matrix, so as to synthesize left invertibility, right invertibility, and dynamic decoupling.

In this paper, we have shown that all of the above extensions can be unified by the study of a particular chain of subspaces naturally associated to the output of a system. As simple corollaries of our analysis, we obtain that right invertibility in the sense of Nijmeijer is the same as in that of Fliess, which in turn is the same as $\rho^* = p$, the number of scalar output components. Left invertibility in the sense of Singh is the same as in that of Fliess, which is the same as $\rho^* = m$, the number of scalar input components. Moreover, the inversion algorithm has a natural interpretation as a procedure for constructing a basis adapted to the chain of subspaces $\mathscr{E}_0 \subset \cdots \subset \mathscr{E}_n$. In a similar vein, the dynamic extension algorithm, which lies at the heart of dynamic decoupling, also constructs a basis adapted to the chain $\mathscr{E}_0 \subset \cdots \subset \mathscr{E}_n$. The main difficulty in comparing the various algorithms was that each was working over its own unique field. This was overcome by relating each of the fields to a common larger field. In this way, the equivalence of four topics, that previously had only been studied separately, was established.

The search for a nonlinear version of the structure at infinity of a linear system has incited much effort on the part of many researchers [2], [7]-[9], [29], with the goal of finding an appropriate tool for solving such classical synthesis problems as noninteracting control and model matching. One of the first efforts in this regard was perceived to possess certain deficiencies [10], because a system could have more zeros at infinity than input or output components, and also because the number of zeros at infinity could be altered by the addition of integrators on the input channels.

Section 3 takes an algebraic approach to defining a nonlinear structure at infinity [9]. Its number of zeros at infinity is always less than or equal to the number of inputs or outputs, and is invariant under the action of regular dynamic feedback. However, the deficiency of this generalization of the structure at infinity is that it cannot properly address the static block noninteracting control problem. Hence the "right" approach, if it exists [30], is yet to be discovered.

REFERENCES

[1] S. N. SINGH, *A modified algorithm for invertibility in nonlinear systems*, IEEE Trans. Automat. Control, AC-26, (1981), pp. 595-598.

[2] H. NIJMEIJER, *Right-invertibility for a class of nonlinear control systems: a geometric approach*, Systems Control Lett., 7 (1986), pp. 125-132.

[3] J. DESCUSSE AND C. H. MOOG, *Dynamic decoupling for right-invertible nonlinear systems*, System Control Lett., 8 (1987), pp. 345-349.

[4] H. NIJMEIJER AND W. RESPONDEK, *Decoupling via dynamic compensation for nonlinear control systems*, in Proc. 25th Conference on Decision and Control, Athens, 1986, pp. 192-197.

[5] M. FLIESS, *A new approach to the noninteracting control problem in nonlinear systems theory*, in Proc. 23rd Allerton Conference, University of Illinois, Monticello, IL, 1985, pp. 123-129.

[6] ———, *A note on the invertibility of nonlinear input-output differential systems*, Systems Control Lett., 8 (1986), pp. 147-151.

[7] ———, *A new approach to the structure at infinity of nonlinear systems*, Systems Control Lett., 7 (1986), pp. 419-421.

[8] H. NIJMEIJER AND J. M. SCHUMACHER, *Zeros at infinity for affine nonlinear control systems*, IEEE Trans. Automat. Control, 30 (1985), pp. 566-573.

[9] C. H. MOOG, *Nonlinear decoupling and structure at infinity*, Math. Control Signals Systems, 1 (1988), pp. 257-268.

[10] A. ISIDORI, *Control of nonlinear systems via dynamic state-feedback*, in Algebraic and Geometric Methods in Nonlinear Control Theory, Proc. Conference Paris, 1985, M. Fliess and M. Hazewinkel, eds., Reidel, Dordrecht, the Netherlands, 1986, pp. 121-145.

[11] J. W. GRIZZLE, M. D. DI BENEDETTO, AND C. H. MOOG, *Computing the differential output rank of a nonlinear system*, in Proc. 26th Conference on Decision and Control, Los Angeles, December 1987, pp. 142-145.

[12] R. ABRAHAM AND J. E. MARSDEN, *Foundations of Mechanics*, Second edtion, Benjamin-Cummings, Reading, May, 1980.

[13] L. M. SILVERMAN, *Inversion of multivariable linear systems*, IEEE Trans. Automat. Control, 14 (1969), pp. 270-276.

[14] R. M. HIRSCHORN, *Invertibility of multivariable nonlinear control systems*, IEEE Trans. Automat. Control, 24 (1979), pp. 855-865.

[15] A. ISIDORI AND C. H. MOOG, *On the equivalent of the notion of transmission zeros*, in Modelling and Adaptive Control, Proc. Mathematical Proc. International Institute of Applied Systems Analysis Conference, Sopron, 1986, C. I. Byrnes and A. Kurszanski, eds., Lecture Notes in Control Information Science, 105, Springer-Verlag, Berlin, New York, 1988.

[16] M. FLIESS, *Some remarks on nonlinear invertibility and dynamic state-feedback*, in Theory and Applications of Nonlinear Control Systems, Mathematical Theory of Networks and Systems 85, Stockholm, C. I. Byrnes and A. Lindquist, eds., North-Holland, Amsterdam, New York, 1986, pp. 115-121.

[17] ———, *Nonlinear control theory and differential algebra*, in Modelling and Adaptive Control, Proc. International Institute of Applied Systems Analysis Conference, Sopron, 1986, C. I. Byrnes and A. Kurszanski, eds., Lecture Notes in Control and Information Science, 105, Springer-Verlag, Berlin, New York, 1988.

[18] J. F. POMMARET, *Géométrie différentielle algébrique et théorie du contrôle*, C.R. Acad. Sci. Paris Sér. I Math., 302 (1986), pp. 547-550.

[19] L. A. RUBEL AND M. F. SINGER, *A differentially algebraic elimination theorem with application to analog computability in the calculus of variations*, Proc. Amer. Math. Soc., 94 (1985), pp. 653-658.

[20] J. F. POMMARET, *Differential Galois Theory*, Gordon and Breach, New York, 1983.

[21] E. R. KOLCHIN, *Differential Algebra and Algebraic Groups*, Academic Press, New York, 1973.

[22] J. JOHNSON, *Kähler differentials and differential algebra*, Ann. of Math., 89 (1969), pp. 92-98.

[23] L. M. SILVERMAN AND A. KITAPÇI, *System structure at infinity*, Systems Control Lett., 3 (1983), pp. 123-131.

[24] C. H. MOOG, *Note on the left-invertibility of nonlinear systems*, in Proc. Mathematical Theory of Networks and Systems 87, Phoenix, AZ, to appear.

[25] A. ISIDORI, A. J. KRENER, C. GORI-GIORGI, AND S. MONACO, *Nonlinear decoupling via feedback: a differential geometric approach*, IEEE Trans. Automat. Control, 26 (1981), pp. 331-345.

[26] J. DESCUSSE, J. F. LAFAY, AND M. MALABRE, *On the structure at infinity of linear block-decouplable systems: the general case*, IEEE Trans. Automat. Control, AC-28 (1983), pp. 1115-1118.

[27] M. FLIESS, *Vers une nouvelle théorie du bouclage dynamique sur la sortie des systèmes non linéaires*, in Analysis and Optimization of Systems, Lecture Notes. Contr. Inform. Sci. 83, Springer-Verlag, Berlin, 1986, pp. 293-299.

[28] S. H. WANG, *Design of precompensator for decoupling problem*, Electronics Letters, 6 (1970), pp. 739-741.

[29] G. CONTE, M. D. DI BENEDETTO, A. ISIDORI, AND A. M. PERDON, *An input-output characterization of the structure at infinity for a nonlinear system*, in Theory and Applications of Nonlinear Control Systems, Mathematical Theory of Networks and Systems 85, Stockholm, C. I. Byrnes and A. Lindquist, eds., North-Holland, Amsterdam, 1986, pp. 301-312.

[30] M. FLIESS, Personal communication, 1986.