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AN OBSERVATION ON THE PARAMETERIZATION OF CAUSAL STABILIZING CONTROLLERS FOR LIFTED SYSTEMS*

J. S. FREUDENBERG[1§] AND J. W. GRIZZLE[1§§]

Abstract. It is shown that the special structure needed to ensure the causality of a controller designed for a lifted periodic/multi-rate finite dimensional discrete-time linear system can be incorporated into a parameterization of the set of stabilizing compensators. A decentralized control problem is also considered.

Key Words—Causality, lifted systems, stabilizing controllers.

1. Introduction

A lot of attention has been focused recently on the analysis and design of periodic digital systems, of periodic controllers for time-invariant systems, and/or systems with multiple sampling rates (see for example, Buescher, 1988; Francis and Georgiou, 1988; Haigawara and Araki, 1988; Khargonekar et al., 1985, and the references therein). One of the main tools used in the above references is the so-called *lifted system* which is essentially a means of constructing a time-invariant representation of a periodic discrete-time system by block processing the inputs and outputs of the system over lengths of time equal to a period of the system (or a multiple thereof). Consider an m-periodic system,

$$\left.\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) \\ y(k) &= C(k)x(k) + E(k)u(k) \end{aligned}\right\}, \tag{1.1}$$

defined for $k \geq 0$, where $x(k) \in \mathscr{R}^n$, $u(k) \in \mathscr{R}^q$, and $y(k) \in \mathscr{R}^p$. Then defining $\zeta(k) = x(mk)$, $\tilde{u}(k) = [u(mk);\cdots;u(m(k+1)-1)]$, and $\tilde{y}(k) = [y(mk);\cdots;y(m(k+1)-1)]$, where the semicolon indicates a new row, results in the time-invariant, *lifted* system,

$$\left.\begin{aligned} \zeta(k+1) &= \tilde{A}\zeta(k) + \tilde{B}\tilde{u}(k) \\ \tilde{y}(k) &= \tilde{C}\zeta(k) + \tilde{E}\tilde{u}(k) \end{aligned}\right\}, \tag{1.2}$$

where the matrices in (1.2) are easily calculated from those of (1.1). For example, when $m=2$, $\tilde{A} = A(1)A(0)$, $\tilde{B} = [A(1)B(0),\ B(1)]$, $\tilde{C} = [C(0);\ C(1)A(0)]$ and $\tilde{E} = [E(0),\ 0;\ C(1)B(0),\ E(1)]$, where once again the semicolon indicates a new row.

When the periodic system arises from multiple sampling rates, or from periodic but non-uniform sampling, a state space representation such as (1.1) is always possible (Berg et al., 1988), but may not be convenient (Buescher, 1988). To fix the ideas, consider the following (conceptual) feedback system.

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Suppose that we have an analog plant whose output is sampled every 1/3 second and fed into a zero order hold (ZOH). The output of the ZOH is then placed in a unity feedback loop around the plant and subtracted from a continuous reference input to form an error signal. The error signal is then sampled every 1/5 second and fed into a ZOH to form the control input signal to the plant. This feedback configuration can be viewed as a periodic system if one inserts samplers at 1/15 [sec.] in front of all the inputs and after all the outputs, and associates states to the zero-order holds. Representing the system first in the form (1.1) and then computing the lifted system would yield a time-invariant system with 15 inputs and 15 outputs. However, one can directly construct a time-invariant model with 5 inputs and 3 outputs (Buescher, 1988). This is mentioned to motivate the slightly more general picture to be used in the next section, which facilitates the application of the main result.

In any case, it is known that these lifting procedures preserve important system properties such as controllability, stabilizability and detectability, and consequently, one can lift the analysis process from the time-varying periodic representation to the time-invariant representation where more tools (CAD or otherwise) are available (Buescher, 1988; Francis and Georgiou, 1988; Haigawara and Araki, 1988; Khargonekar et al., 1985). In principle, one can also lift the controller design process as long as one keeps in mind certain constraints imposed by *causality* (Khargonekar et al., 1985): since $u(k_1)$ cannot be allowed to depend upon $y(k_2)$ for $k_2>k_1$, this means that various components of $\tilde{u}(\tilde{k}_1)$ cannot be allowed to depend upon certain components of $\tilde{y}(\tilde{k}_1)$, but there are no constraints on how $\tilde{u}(\tilde{k}_1)$, may depend upon $\tilde{y}(\tilde{k}_2)$ for $\tilde{k}_2<\tilde{k}_1$. However, as long as a controller designed on the basis of the lifted system satisfies such causality constraints, it can be directly implemented on the underlying system (1.1) because the operations of forming $\tilde{y}$ from y and obtaining u from $\tilde{u}$ are just the operations of demultiplexing and multiplexing respectively; in other words, just a question of data handling.

While this note was in preparation, the authors received a preprint from Meyer (1988) which treats the problem addressed in this note through a generalized notion of shift invariance. The present approach allows one to consider any sampling scheme which leads to a time-invariant system after appropriate block processing of the inputs and outputs.

2. Main Result

Suppose that the system (1.2) is a time-invariant (lifted) representation of a causal periodic system with *period* T seconds (T need not be the least period of the system). It is convenient to abuse notation and confuse $\tilde{y}(k)$ with $\tilde{y}(kT)$ and similarly $\tilde{u}(k)$ with $\tilde{u}(kT)$ so that time is now explicit. Then, each component of $\tilde{u}$ can be written as $\tilde{u}_j(kT)=u_{l(j)}(kT+\alpha_j)$ for a least $0\leq\alpha_j<T$, and some $l(j)$ in the integer set $\{1,\cdots,q\}$ and similarly $\tilde{y}_i(kT)=y_{h(i)}(kT+\beta_i)$ for a least $0\leq\beta_i<T$, and some $h(i)$ in the integer set $\{1,\cdots,p\}$. Then, for a controller $K(z)$ to be causal, it is necessary and sufficient that it satisfies $K_{ji}(\infty)=0$, whenever $\beta_i>\alpha_j$; that is the $(j,\ i)$-component must be a strictly proper transfer function (it is assumed here that computation time, if non-negligible, has been taken into account through the addition of delays in the original model). Note that the condition that the plant (1.2) is causal can be stated as $\tilde{P}_{ij}(\infty)=0$ whenever $\beta_i<\alpha_j$, where $\tilde{P}$ is its transfer function, assumed to be $\tilde{p}\times\tilde{q}$.

To fix the notation, assume that $p=q=2$ and that $\tilde{u}(kT)=[u_1(kT);\ u_2(kT);\ u_1(kT+T/2);\ u_2(kT+T/2)]$. Then, $l(1)=l(3)=1$, $l(2)=l(4)=2$, $\alpha_1=\alpha_2=0$ and $\alpha_3=\alpha_4=T/2$. Suppose further that $\tilde{y}(kT)=[y_1(kT);\ y_2(kT);\ y_2(kT+T/2);\ y_1(kT+2T/3)]$. Then, $h(1)=h(4)=1$, $h(2)=h(3)=2$, $\beta_1=\beta_2=0$, $\beta_3=T/2$ and $\beta_4=2T/3$. Hence, a controller $K(z)$ will be causal, if and only if, $K_{13}(\infty)=K_{14}(\infty)=K_{23}(\infty)=K_{24}(\infty)=K_{34}(\infty)=K_{44}(\infty)=0$.

The main observation is that such a constraint on the class of controllers can be incorporated into a parameterization of the set of all stabilizing controllers analogous to those found in Desoer et al. (1980), Kucera (1979), Vidyasagar (1985) and Youla et al. (1976). Such parameterizations describe the set of stabilizing controllers in terms of a free parameter which may be a polynomial matrix (Kucera, 1979; Youla, et al., 1976) or a stable transfer function (Desoer et al., 1980; Vidyasagar, 1985) depending on the context. (The free parameter is sometimes termed the YJBK parameter after the authors of Youla et al. (1976) and Kucera (1979).) We shall show that this procedure may be extended to incorporate the constraint of causality. Our construction will utilize the elements of a doubly coprime factorization of the plant $\tilde{P}(z)$ with state description (1.2).

For brevity, we merely present the formulas devised by Nett et al. (1984) (see also Vidyasagar, 1985, p. 83) for constructing such a factorization; the reader is referred to these sources for definitions and further explanation.

Lemma (Nett et al., 1984). Consider the transfer function $\tilde{P}(z)$ with realization (1.2), and suppose that $(\tilde{A}, \tilde{B})$ and $(\tilde{A}, \tilde{C})$ are, respectively, stabilizable and detectable. Choose constant matrices G and F such that $A_r \triangleq \tilde{A} - \tilde{B}G$ and $A_l \triangleq \tilde{A} - F\tilde{C}$ have stable eigenvalues[†]. Then, $\tilde{P} = N_r D_r^{-1} = D_l^{-1} N_l$ and

$$\begin{bmatrix} Y_r & X_r \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & -X_l \\ N_r & Y_l \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

where

$$\begin{aligned} N_l &= \tilde{C}(zI - A_l)^{-1}(\tilde{B} - F\tilde{E}) + \tilde{E}, \\ D_l &= I - \tilde{C}(zI - A_l)^{-1}F, \\ N_r &= (\tilde{C} - \tilde{E}G)(zI - A_r)^{-1}\tilde{B} + \tilde{E}, \\ D_r &= I - G(zI - A_r)^{-1}\tilde{B}, \\ X_r &= G(zI - A_l)^{-1}F, \\ Y_r &= I + G(zI - A_l)^{-1}(\tilde{B} - F\tilde{E}), \\ X_l &= G(zI - A_r)^{-1}F, \\ Y_l &= I + (\tilde{C} - \tilde{E}G)(zI - A_r)^{-1}F. \end{aligned}$$

Theorem. Suppose that the plant (1.2) is causal, stabilizable and detectable. Let (N_r, D_r), (N_l, D_l), X_r, Y_r, X_l and Y_l be as defined in the preceding Lemma. Then, the set of all causal linear time invariant finite dimensional stabilizing compensators is characterized in terms of a free parameter Q as the set,

$$\{(Y_r - QN_l)^{-1}(X_r + QD_l) \colon Q \in M(z),\ |Y_r - QN_l| \neq 0\}, \tag{2.1}$$

where $M(z)$ is the set of all stable $\tilde{q} \times \tilde{p}$ matrices of rational functions such that $Q_{ji}(\infty) = 0$ whenever $\beta_i > \alpha_j$.

Proof. From Vidyasagar (1985), it is known that $K(z) \triangleq (Y_r - QN_l)^{-1}(X_r + QD_l)(z)$, $Q \in M(z)$, yields a stabilizing compensator, so it remains to show that K is causal in the sense discussed earlier. By the Lemma, one has that $N_l(\infty) = \tilde{P}(\infty)$, $D_l(\infty) = \tilde{p} \times \tilde{p}$ identity matrix, $X_r(\infty) = 0$ and $Y_r(\infty) = \tilde{q} \times \tilde{q}$ identity matrix. Then, it is a straightforward matter to evaluate $K(z)$ at infinity and see that it satisfies the condition for causality. For example, consider the transfer function $QN_l(z)$, which maps inputs into inputs. To see that it is causal, define a causality relation R_{io} from inputs to outputs by the jth output is

[†] For the present purpose, it suffices to take the region of stability as the open unit disk in the z-plane, although more general stability regions may also be treated in this framework (Nett et al., 1984).

related to the ith input, if and only if $\beta_j - \alpha_i \geq 0$, if and only if $\tilde{P}_{ji}(\infty)$ may be nonzero; similarly define a causality relation R_{OI} from outputs to inputs by the kth input is related to the jth output, if and only if $\alpha_k - \beta_j \geq 0$, if and only if $Q_{kj}(\infty)$ may be nonzero. Consider now the composite relation $R_{II} = R_{IO} R_{OI}$. From Ross and Wright (1985, p. 279 and p. 290), the kth input component is related to the ith input component, if and only if there exists j' such that $\alpha_k - \beta_{j'} \geq 0$ and $\beta_{j'} - \alpha_i \geq 0$; but this is equivalent to $\alpha_k \geq \alpha_i$, which establishes causality. The rest of the sufficiency is similar and is left to the reader; the fact that every causal stabilizing compensator is given by (2.1) follows as in Vidyasagar (1985, p. 108).

Continuing with the example considered earlier in the Section, we have that *any* 4×4 rational matrix $Q(z)$ with $Q_{13}(\infty) = Q_{14}(\infty) = Q_{23}(\infty) = Q_{24}(\infty) = Q_{34}(\infty) = Q_{44}(\infty) = 0$ will yield a causal compensator $K(z)$. If, in addition, $Q(z)$ is stable, then $K(z)$ will be a causal stabilizing compensator.

Remark: The above result allows the interesting design technique of Boyd et al. (1988) to be immediately applied to periodic/multi-rate systems, since the constraints on the YJBK parameter are linear. The linearity of the constraints may also make it possible to modify the results of Dahleh and Pearson (1987) to allow the treatment of periodic/multi-rate systems.

3. Comments

There are other examples of special structure that can be incorporated into the YJBK parameterization. Consider a decentralized linear discrete-time system of the form,

$$\left.\begin{aligned} x(k+1) &= Ax(k) + \sum_{i=1}^{p} B_i u_i(k), \\ y_j(k) &= C_j x(k) + E_j u_j(k), \qquad 1 \leq j \leq p \end{aligned}\right\}, \tag{3.1}$$

and suppose that one is allowing a one-step delay sharing information pattern (Hsu and Marcus, 1982); that is, at time k, the ith controller has available $y_i(k)$ and the entire measurement vector $y(l)$ for $l<k$. For simplicity of notation, let u_i and y_i be scalar valued. Then, the transfer function $P(z)$ has the property that $P_{ij}(\infty)=0$ for $i \neq j$ and an admissible controller must satisfy the same constraint. It is straightforward to check that (2.1), with $M(z)$ being the set of stable $p \times p$ matrices with rational entries such that $Q \in M(z)$ implies that $Q_{ij}(\infty)=0$ whenever $i \neq j$, parameterizes the set of all admissible causal linear finite-dimensional stabilizing compensators.

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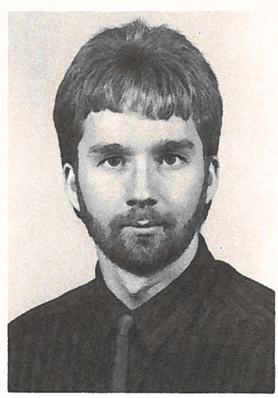

methods, and multirate digital systems.

Dr. Grizzle was a (full) Fulbright Grant awardee and a NATO Postdoctoral Fellow from January to December 1984. He also received a Presidential Young Investigator Award.