Remarks on Event-Based Stabilization of Periodic Orbits in Systems with Impulse Effects

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***Abstract*— This note focuses on control design issues associated with creating and stabilizing periodic orbits in systems with impact events via control decisions that take place only at event-triggered transitions in the dynamics. Many hybrid, underactuated, mechanical systems with impacts arising in legged locomotion can be modeled as systems with impulse effects. Constructive techniques are needed for the design and analysis of periodic orbits in such systems.**

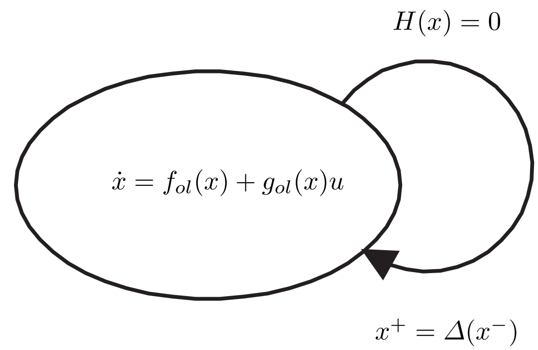

Fig. 1. A nonlinear system with impulse effects. In the case of a bipedal robot, the state consists of the generalized configuration and velocity variables, the transition condition corresponds to the height of the swing foot above the ground going to zero, and the re-initialization rule corresponds to the impact map of two rigid bodies (the swing foot and the ground). In this note, it will be assumed that a parameterized family of feedback laws $u(x, a)$, $a \in \mathcal{A}$, has been designed for the continuous phase so that $f(x, a) = f_{ol}(x) + g_{ol}(x)u(x, a)$. The problem of determining event-based update laws for the parameter a is investigated for the existence of asymptotically stable periodic orbits.

I. Introduction

Mechanical legged locomotion is being studied for its enhanced maneuverability in rough terrain, for its ability to deal with environments with discontinuous supports, such as the rungs of a ladder, and because of the popular appeal of machines that operate in anthropomorphic ways[1]. The dynamic models that arise in the study of legged locomotion are fundamentally hybrid. For example, a bipedal walking motion consists of successive phases of single support (meaning the stance leg is touching the ground and the swing leg is not) and double support (both legs are in contact with the ground), while running consists of successive phases of single support and flight (there is no contact with the ground). It is common to model the impact that occurs when the swing leg strikes the ground as an instantaneous contact of rigid bodies, which results in an algebraic representation of the impact event; see Figure 1.

A canonical problem in legged robots is how to design a controller that generates closed-loop motions, such as walking or running, that are periodic and stable (i.e., limit cycles). Due to the hybrid nature of the system, this task is far from being solved through existing control methods. New paradigms, concepts, and control analysis techniques are thus needed to deal with this class of systems [4]. Because the system model is hybrid, it is natural for the controller to be hybrid as well, with control actions taking place during the continuous phase(s) [7], [16] as well as at discrete transitions [18], [8], [17], [5]. This note focuses on the latter issue.

The method of Poincaré sections and return maps is widely used to determine the existence and stability of periodic orbits in a broad range of system models, such as time-invariant and periodically-time-varying ordinary differential equations [14], [9], hybrid systems consisting of several time-invariant ordinary differential equations linked by event-based switching mechanisms and re-initialization rules [7], [13], [15], [12], differential-algebraic equations [10], and relay systems with hysteresis [6]. The conceptual advantage of the method of Poincaré is that it reduces the study of periodic orbits to the study of equilibrium points of sampled-data systems, with the latter being a more extensively studied problem. In particular, this note is based on the fact that parameter values that are held constant within the continuous phase of the hybrid dynamics and updated at impact events appear as standard sampled-data controls in the Poincaré return map. This observation leads to the formulation of sampled-data feedback design problems for the creation and stabilization of periodic orbits in nonlinear systems with impulse effects.

II. Background

This section reviews the notions of systems with impulse effects, periodic orbits, stability of periodic orbits, the method of Poincaré and hybrid invariance. The objective is to set the stage for studying event-based control actions in Section III.

A. Systems with impulse effects

An autonomous system with impulse effects consists of an autonomous ordinary differential equation,

$$\dot{x} = f(x), \tag{1}$$

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[1] In this regard, Honda has just announced that Asimo is performing the job of a receptionist in an office setting [1].

defined on some state space $\mathcal{X}$, a co-dimension one surface $\mathcal{S} \subset \mathcal{X}$ at which solutions of the differential equation undergo a discrete transition that is modeled as an instantaneous re-initialization of the differential equation, and a rule $\Delta : \mathcal{S} \to \mathcal{X}$ that specifies the new initial condition as a function of the point at which the solution impacts $\mathcal{S}$ [2], [19]. The co-dimension one surface $\mathcal{S}$ is called a transition surface or an impact surface (in analogy with biped models) and Δ is often called an impact map. The system will be denoted by

$$\Sigma : \begin{cases} \dot{x} &= f(x) \qquad x^- \notin \mathcal{S} \\ x^+ &= \Delta(x^-) \quad x^- \in \mathcal{S}, \end{cases} \tag{2}$$

and the following hypotheses will be made:

HSH0) $\mathcal{X}$ is a smooth embedded submanifold of $\mathbb{R}^n$;
HSH1) $f : \mathcal{X} \to T\mathcal{X}$ is continuous, and a solution of $\dot{x} = f(x)$ from a given initial condition is unique and depends continuously on the initial condition;
HSH2) there exist a differentiable function $H : \mathcal{X} \to \mathbb{R}$, such that $\mathcal{S} = \{x \in \mathcal{X} \mid H(x) = 0\}$; moreover, for every $s \in \mathcal{S}$, $\frac{\partial H}{\partial x}(s) \neq 0$.
HSH3) $\Delta : \mathcal{S} \to \mathcal{X}$ is continuous, where $\mathcal{S}$ is given the subset topology from $\mathcal{X}$.
HSH4) $\overline{\Delta(\mathcal{S})} \cap \mathcal{S} = \emptyset$, where $\overline{\Delta(\mathcal{S})}$ is the set closure of $\Delta(\mathcal{S})$.

In simple terms, a solution of (2) is specified by the differential equation (1) until its state "impacts" the hyper surface $\mathcal{S}$ at some time t_I. At t_I, the impulse model Δ compresses the impact event into an instantaneous moment of time, resulting in a discontinuity in the state trajectory. The impact model provides the new initial condition from which the differential equation evolves until the next impact with $\mathcal{S}$. In order to avoid the state having to take on two values at the "impact time" t_I, the impact event is, roughly speaking, described in terms of the values of the state "just prior to impact" at time "t_I^-", and "just after impact" at time "t_I^+". These values are represented by x^- and x^+, respectively.

From this description, a formal definition of a solution is easily written down by piecing together appropriately initialized solutions of (1); see [19], [7], [13], [3]. A choice must be made as to whether to take a solution $\varphi(t)$ of (2) as being a left- or right-continuous function of time at each impact event; here, solutions are assumed to be right continuous [7].

B. Periodic orbits

A solution $\varphi(t)$ of (2) is *periodic* if there exists a finite $T > 0$ such that $\varphi(t + T) = \varphi(t)$ for all $t \in [t_0, \infty)$. A set $\mathcal{O} \subset \mathcal{X}$ is a *periodic orbit* of (2) if $\mathcal{O} = \{\varphi(t) \mid t \geq t_0\}$ for some periodic solution $\varphi(t)$. While a system with impulse effects may certainly have periodic solutions that do not involve impact events, they are not of interest here because they could be studied more simply as solutions of (1). If a periodic solution has an impact event, then the corresponding periodic orbit $\mathcal{O}$ is not closed; see [7] and Fig. 2. Let $\bar{\mathcal{O}}$ denote its set closure.

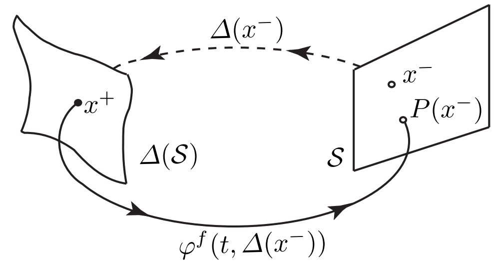

Fig. 2. Geometric interpretation of a Poincaré return map $P : \mathcal{S} \to \mathcal{S}$ for a system with impulse effects. The Poincaré section is selected as the switching surface, $\mathcal{S}$. A periodic orbit exists when $P(x^-) = x^-$. Due to right-continuity of the solutions, x^- is not an element of the orbit. With left-continuous solutions, $\Delta(x^-)$ would not be an element of the orbit.

Notions of stability in the sense of Lyapunov, and asymptotic stability of orbits follow the standard definitions; see [11, pp. 302], [7], [13]. A periodic orbit $\mathcal{O}$ is *transversal* to $\mathcal{S}$ if its closure intersects $\mathcal{S}$ in exactly one point, and for $\bar{x} := \bar{\mathcal{O}} \cap \mathcal{S}$, $L_f H(\bar{x}) := \frac{\partial H}{\partial x}(\bar{x}) f(\bar{x}) \neq 0$ (in words, at the intersection, $\bar{\mathcal{O}}$ is not tangent to $\mathcal{S}$, where $\bar{\mathcal{O}}$ is the set closure of $\mathcal{O}$).

C. Poincaré return map

In order to study periodic orbits with impact events, it is natural to select $\mathcal{S}$ as the Poincaré section. To define the return map, let $\varphi^f(t, x_0)$ denote the maximal solution of (1) with initial condition x_0 at time $t_0 = 0$. The *time-to-impact* function, $T_I : \mathcal{X} \to I\!R \cup \{\infty\}$, is defined by

$$T_I(x_0) := \begin{cases} \inf\{t \geq 0 | \varphi^f(t, x_0) \in \mathcal{S}\} & \text{if } \exists\, t \text{ such that} \\ & \varphi(t, x_0) \in \mathcal{S} \\ \infty & \text{otherwise.} \end{cases} \tag{3}$$

The Poincaré return map, $P : \mathcal{S} \to \mathcal{S}$, is then given as (the partial map)

$$P(x) := \varphi^f(T_I \circ \Delta(x), \Delta(x)). \tag{4}$$

Remark: From [7], P is well defined on the open set $\tilde{\mathcal{S}} := \Delta^{-1}(\tilde{\mathcal{X}})$, where

$$\tilde{\mathcal{X}} = \{x \in \mathcal{X} \mid 0 < T_I(x) < \infty,\ L_f H(\varphi^f(T_I(x), x)) \neq 0\}.$$

For many reasons, it is more convenient to work with P as a partial map.

Theorem 1: (Method of Poincaré Sections for Systems with Impulse Effects) [7] Under HSH0)–HSH4), the following statements hold:

a) If $\mathcal{O}$ is a periodic orbit of (2) that is transversal to $\mathcal{S}$, then there exists a point $x^* \in \mathcal{S}$ such that $L_f H(x^*) \neq 0$ and $\Delta(x^*)$ generates $\mathcal{O}$, that is $\mathcal{O} = \{\varphi^f(t, \Delta(x^*)) \mid 0 \leq t < T_I \circ \Delta(x^*)\}$.
b) $x^* \in \mathcal{S}$ is a fixed point of P and $L_f H(x^*) \neq 0$ if, and only if, $\Delta(x^*)$ generates a periodic orbit that is transversal to $\mathcal{S}$.
c) $x^* \in \mathcal{S}$ is a stable equilibrium point of $x[k+1] = P(x[k])$ and $L_f H(x^*) \neq 0$ if, and only if, the orbit $\mathcal{O}(\Delta(x^*))$ is stable and transversal to $\mathcal{S}$.

d) $x^* \in \mathcal{S}$ is an asymptotically stable equilibrium point of $x[k+1] = P(x[k])$ and $L_f H(x^*) \neq 0$ if, and only if, the orbit $\mathcal{O}(\Delta(x^*))$ is asymptotically stable and transversal to $\mathcal{S}$.

D. Invariance conditions and attractivity

A set $Z \subset \mathcal{X}$ is *forward invariant* if for each $x_0 \in Z$, there exists $t_1 > 0$ such that $\varphi^f(t, x_0) \in Z$ for $t \in [0, t_1)$. Z is *impact invariant* if $\mathcal{S} \cap Z \neq \emptyset$, and $\Delta(\mathcal{S} \cap Z) \subset Z$. Z is *hybrid invariant* if it is both forward invariant and impact invariant.

Define the *settling time to* Z, $T_Z^{\rm set} : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$, by

$$T_Z^{\rm set}(x_0) := \begin{cases} \inf\{\tau \geq 0 | \exists \tau_1 > \tau, \varphi^f(t, x_0) \in Z,\ t \in [\tau, \tau_1)\} \\ \qquad\qquad \text{if } \exists\ t \text{ such that } \varphi^f(t, x_0) \in Z \\ \infty \quad \text{otherwise.} \end{cases} \tag{5}$$

Z is *locally continuously finite-time attractive* if Z is forward invariant and there exists an open set $\mathcal{V}$ containing Z such that $T_Z^{\rm set}$ is finite and continuous at each point of $\mathcal{V}$.

E. Restricted Poincaré map

It is advantageous to analyze the autonomous system with impulse effects (2) when it possesses a subset $Z \subset \mathcal{X}$ satisfying the hypotheses below.

InvH1) Z is an embedded submanifold of $\mathcal{X}$.
InvH2) $\mathcal{S} \cap Z$ is an embedded submanifold with dimension one less than the dimension of Z.
InvH3) Z is locally continuously finite-time attractive.
InvH4) Z is hybrid invariant (forward invariant and impact invariant).

By forward invariance, solutions of $\dot{x} = f(x)$ initialized in Z remain in Z. Denote the restriction of f to Z by $f_{|Z}$ and the associated differential equation by $\dot{z} = f_{|Z}(z)$. Similarly, let $H_{|Z}$ and $\Delta_{|\mathcal{S}\cap Z}$ denote the restriction of H and Δ to Z. We note that Hypotheses HSH0)–HSH1) on (2) imply the corresponding properties on the restriction dynamics. Indeed, $H_{|Z}$ clearly satisfies HSH2), and by impact invariance, $\Delta_{|\mathcal{S}\cap Z} : \mathcal{S} \cap Z \to Z$ by $\Delta_{|\mathcal{S}\cap Z}(z) := \Delta(z)$, $z \in Z$, satisfies HSH3) and HSH4). Hence, the hybrid restriction dynamics

$$\Sigma_{|Z} : \begin{cases} \dot{z} = f_{|Z}(z) & z^- \notin \mathcal{S} \cap Z \\ z^+ = \Delta_{|\mathcal{S}\cap Z}(z^-) & z^- \in \mathcal{S} \cap Z \end{cases} \tag{6}$$

is a system with impulse effects in its own right, verifying Hypotheses HSH0)–HSH4) with respect to its state space, Z. Therefore, Theorem 1 on the method of Poincaré sections can be applied to characterize periodic orbits in (6). In order to profitably use this observation, two further observations need to be made: (1) By construction, periodic orbits of the hybrid restriction dynamics are also periodic orbits of the full-order model (2). (2) Let $\rho : Z \to Z$ denote the Poincaré map of the hybrid restriction dynamics. Then $\rho = P_{|Z}$. This next result establishes conditions under which the stability properties of orbits of the hybrid restriction dynamics carry over to the full-order dynamics. In other words, the properties of certain periodic orbits of the full-order dynamic can be completely determined on the basis of a lower order model.

Theorem 2: (Poincaré for the Hybrid Restriction Dynamics)[7], [16] Assume that the autonomous system with impulse effects, (2), satisfies Hypotheses HSH0)–HSH4). Suppose furthermore that $Z \subset \mathcal{X}$ satisfies InvH1)–InvH4). Then (2) has a stable (resp., asymptotically stable) orbit transversal to $\mathcal{S} \cap Z$ if, and only if, the discrete-time system

$$x[k+1] = \rho(x[k]) \tag{7}$$

with state space $\mathcal{S} \cap Z$ has a stable (resp., asymptotically stable) equilibrium point x^* such that $L_f H(x^*) \neq 0$.

III. Event-Based Control

In this section, we assume that various elements of the system with impulse effects (2) depend on one or more parameters that are to be held constant between transition events, but at each transition, the parameters may be updated. This situation arises, for example, when a within-stride controller for a bipedal robot has been designed to depend on a (possibly vector valued) parameter in such a way that by changing the parameter's value, different locomotion characteristics may be achieved, such as walking at a different speed, or with a different step length; see [18], [8], [17], [5]. The parameter will be assumed to take on a continuum of values and may be updated at each transition event. The objective is to analyze when a given event-based update rule for the parameter will result in an asymptotically stable, periodic orbit for the system with impulse effects.

A. Analyzing Event-Based Control with the Full-Order Model

Consider a collection of systems with impulse effects, indexed by a parameter a,

$$\Sigma_a : \begin{cases} \dot{x} = f(x, a) & x^- \notin \mathcal{S} \\ x^+ = \Delta(x^-, a) & x^- \in \mathcal{S}, \end{cases} \tag{8}$$

with common state space $x \in \mathcal{X}$ and impact set $\mathcal{S}$, and suppose that Hypotheses HSH0) and HSH2)–HSH4) hold. Assume that a takes values in $\mathcal{A}$, an open subset of $\mathbb{R}^p$, and that Hypothesis HSH1 is strengthened to hold for the associated differential equation

$$\begin{aligned} \dot{x} &= f(x, a) \\ \dot{a} &= 0, \end{aligned} \tag{9}$$

that is, f is continuous on $\mathcal{X} \times \mathcal{A}$ and solutions of (9) exist, are unique, and depend continuously on initial conditions.

For $a \in \mathcal{A}$, let $P_a : \mathcal{S} \to \mathcal{S}$ be the Poincaré return map of (8). However, instead of considering the difference equation $x[k+1] = P_a(x[k])$ on $\mathcal{S}$, we now invoke the fact that a can be changed at each impact and we view the difference equation as a discrete-time control system on $\mathcal{S}$ with the parameter vector $a \in \mathcal{A}$ as the control:

$$x[k+1] = P(x[k], a[k]), \tag{10}$$

where $P(x,a) := P_a(x)$. It will now be established that there is a one-to-one correspondence between static (resp., dynamic) state-variable feedback control laws for (10) and static (resp., dynamic) parameter update laws for (8). Moreover, thanks to Poincaré analysis, this correspondence extends to periodic orbits and their stability properties.

Theorem 3: (Stability under Event-Based Parameter Updates-I) Consider the collection of systems with impulse effects, (8), with $a \in \mathcal{A}$, an open subset of $\mathbb{R}^p$. Suppose that $\mathcal{X}$ and $\mathcal{S}$ satisfy Hypotheses HSH0), HSH2)–HSH4). Suppose furthermore that Hypothesis HSH1) holds for the differential equation (9). Let $\mathcal{W}$ be an open subset of $\mathbb{R}^\ell$ for some integer ℓ, and define $\mathcal{X}_{aux} := \mathcal{X} \times \mathcal{A} \times \mathcal{W}$ and $\mathcal{S}_{aux} := \mathcal{S} \times \mathcal{A} \times \mathcal{W}$. Suppose that $v_1 : \mathcal{S} \times \mathcal{W} \to \mathcal{A}$ and $v_2 : \mathcal{S} \times \mathcal{W} \to \mathcal{W}$ are continuous. Then,

$$\begin{bmatrix} \dot{x} \\ \dot{a} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} f(x,a) \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} x^- \\ a^- \\ w^- \end{bmatrix} \notin \mathcal{S}_{aux}$$

$$\begin{bmatrix} x^+ \\ a^+ \\ w^+ \end{bmatrix} = \begin{bmatrix} \Delta(x^-, v_1(x^-, w^-)) \\ v_1(x^-, w^-) \\ v_2(x^-, w^-) \end{bmatrix} \quad \begin{bmatrix} x^- \\ a^- \\ w^-) \end{bmatrix} \in \mathcal{S}_{aux}, \tag{11}$$

has a stable (resp., asymptotically stable) orbit transversal to $\mathcal{S}_{aux}$ if, and only if, the discrete-time system

$$\begin{aligned} x[k+1] &= P(x[k], v_1(x[k], w[k])) \\ w[k+1] &= v_2(x[k], w[k]) \end{aligned} \tag{12}$$

on $\mathcal{S} \times \mathcal{W}$ has a stable (resp., asymptotically stable) equilibrium point $(x^*; w^*)$ such that $L_f H(x^*, a^*) \neq 0$, where $a^* = v_1(x^*, w^*)$.

Proof: Suppose first that $a = v(x)$ is a static state-variable feedback control law for (10) and consider the discrete-time closed-loop system

$$x[k+1] = P(x[k], v(x[k])), \tag{13}$$

and a deadbeat dynamic extension

$$\begin{aligned} x[k+1] &= P(x[k], v(x[k])) \\ a[k+1] &= v(x[k]). \end{aligned} \tag{14}$$

Note that (13) has an equilibrium point if, and only if, (14) has an equilibrium point, and moreover, x^* is stable (resp., asymptotically stable) equilibrium point for (13) if, and only if, $(x^*; a^* = v(x^*))$ is a stable (resp., asymptotically stable) equilibrium point for (14). The importance of this observation is that

$$P_{aux}(x,a) := \begin{bmatrix} P(x, v(x)) \\ v(x) \end{bmatrix} \tag{15}$$

is the Poincaré return map of the following system with impulse effects:

$$\begin{bmatrix} \dot{x} \\ \dot{a} \end{bmatrix} = \begin{bmatrix} f(x,a) \\ 0 \end{bmatrix} \qquad \begin{bmatrix} x^- \\ a^- \end{bmatrix} \notin \mathcal{S}_{aux}$$

$$\begin{bmatrix} x^+ \\ a^+ \end{bmatrix} = \begin{bmatrix} \Delta(x^-, v(x^-)) \\ v(x^-) \end{bmatrix} \qquad \begin{bmatrix} x^- \\ a^- \end{bmatrix} \in \mathcal{S}_{aux}, \tag{16}$$

where the state space is $\mathcal{X}_{aux} := \mathcal{X} \times \mathcal{A}$ and the impact surface is $\mathcal{S}_{aux} := \mathcal{S} \times \mathcal{A}$. Hence, by Theorem 1, designing a memoryless parameter-update law for (8) that results in (16) possessing a stable (resp., asymptotically stable) periodic orbit is precisely equivalent to designing a static state-feedback control law for (10) that results in (13) possessing a stable (resp., asymptotically stable) equilibrium point. Since the same reasoning applies mutatis mutandis for the more general case of a parameter update law with memory (i.e., a dynamic event-based feedback controller), the result is proved. ■

Remark: The special case of a memoryless parameter update for (8), and hence, static state feedback control of (10), is obtained by letting $\mathcal{W}$ be empty. Integral feedback control action, either to reject a constant disturbance or to track a constant reference, is also a special case. If d and r are constants (possibly vector valued) representing disturbances and references, respectively, then formally define $f(x,a) = \tilde{f}(x,a,d)$, $v_1(x,w) = \tilde{v}_1(x,w,r)$ and $v_2(x,w) = \tilde{v}_2(x,w,r)$ in the above analysis.

Remark: In words, Theorem 3 states that the design of a parameter update law for (8) that creates an asymptotically stable periodic orbit can be performed by designing a feedback controller for (10) that creates an asymptotically stable equilibrium point. Even more specifically, suppose there exists a parameter value a^* for which (8) possesses a desired periodic orbit, but the orbit is either not stable, or it is asymptotically stable, but the rate of convergence is too slow. Let x^* be the corresponding fixed point of P_{a^*}. Then designing a parameter update law for (8) that preserves the orbit and stabilizes it (or increases the rate of convergence) is equivalent to designing a feedback controller for (10) that preserves the equilibrium point and stabilizes it (or increases the rate of convergence).

B. Analyzing Event-Based Actions with a Hybrid Restriction Dynamics and Finite-Time Attractivity

The previous subsection reduced the study of orbits in a collection of systems with impulse effects, having a common state space and a common impact surface, to the study of equilibrium points of a discrete-time control system evolving on the impact surface. This subsection will identify circumstances in which analysis and feedback controller design for the discrete-time control system can be performed on the restriction dynamics, thereby reducing the dimension of the feedback design problem; for concrete examples, see [18], [8], [17], [5].

We present two refinements of Theorem 3 to allow the event-based feedback design to be performed on the restriction dynamics. Consider a collection of subsets $\{Z_a \mid a \in \mathcal{A}\} \subset \mathcal{X}$. In the first case, we suppose that $\mathcal{S} \cap Z_a$ is independent of $a \in \mathcal{A}$. We denote the common intersection by $\mathcal{S} \cap Z_\diamondsuit$. Under this assumption, hybrid invariance leads to a restricted Poincaré map, $\rho_a : \mathcal{S} \cap Z_\diamondsuit \to \mathcal{S} \cap Z_\diamondsuit$. Under appropriate hypotheses, the reduction method of Theorem 2 can be combined with Theorem 3 so that event-based feedback design can be carried out on the control system $x[k+1] = \rho(x[k], a[k])$ evolving on the state space $\mathcal{S} \cap Z_\diamondsuit$ with controls taking values in $\mathcal{A}$.

Theorem 4: (Stability under Event-Based Parameter Updates-II) Consider the collection of systems with impulse effects, (8), with the parameter a taking values in $\mathcal{A}$. Suppose that $\mathcal{X}$ and $\mathcal{S}$ satisfy Hypotheses HSH0) and HSH2)–HSH4). Suppose furthermore that $\mathcal{A}$ is an open subset of $\mathbb{R}^p$ such that Hypothesis HSH1) holds for the differential equation (9) and there exists a collection of subsets $\{Z_a \mid a \in \mathcal{A}\} \subset \mathcal{X}$ such that:

1) $\forall a \in \mathcal{A}$, $Z_a \subset \mathcal{X}$ satisfies Hypotheses InvH1) and InvH2);
2) $\forall a \in \mathcal{A}$, $\mathcal{S} \cap Z_a$ is independent of a; denote the common intersection with $\mathcal{S}$ by $\mathcal{S} \cap Z_\diamondsuit$;
3) $\forall a \in \mathcal{A}$, $\Delta(\mathcal{S} \cap Z_\diamondsuit, a) \subset Z_a$.
4) $\mathbf{Z} := \{(x, a) \mid x \in Z_a, a \in \mathcal{A}\}$ is an embedded submanifold of $\mathcal{X} \times \mathcal{A}$ and is locally continuously finite-time attractive for (9).

Let $\mathcal{W}$ be an open subset of $\mathbb{R}^\ell$ suppose that $v_1 : \mathcal{S} \times \mathcal{W} \to \mathcal{A}$ and $v_2 : \mathcal{S} \times \mathcal{W} \to \mathcal{W}$ are given continuous maps. Define $\mathcal{X}_{aux} := \mathcal{X} \times \mathcal{A} \times \mathcal{W}$, $\mathcal{S}_{aux} := \mathcal{S} \times \mathcal{A} \times \mathcal{W}$, and $\mathcal{Z}_{aux} := \mathbf{Z} \times \mathcal{W}$. Then (11) has a stable (resp., asymptotically stable) orbit transversal to $\mathcal{S}_{aux} \cap \mathcal{Z}_{aux}$ if, and only if, the discrete-time system

$$\begin{aligned} x[k+1] &= \rho(x[k], v_1(x[k], w[k])) \\ w[k+1] &= v_2(x[k], w[k]) \end{aligned} \tag{17}$$

on $\mathcal{S} \cap Z_\diamondsuit \times \mathcal{W}$ has a stable (resp., asymptotically stable) equilibrium point $(x^*; w^*)$ such that $L_f H(x^*, a^*) \neq 0$, where $a^* = v_1(x^*, w^*)$. ♦

Proof: For clarity, first assume that $\mathcal{W} = \emptyset$ and consider

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{a} \end{bmatrix} &= f_{aux}(x, a) \qquad & \begin{bmatrix} x^- \\ a^- \end{bmatrix} \notin \mathcal{S}_{aux} \\ \begin{bmatrix} x^+ \\ a^+ \end{bmatrix} &= \Delta_{aux}(x^-, a^-) \qquad & \begin{bmatrix} x^- \\ a^- \end{bmatrix} \in \mathcal{S}_{aux}, \end{aligned} \tag{18}$$

where the state space is $\mathcal{X}_{aux} := \mathcal{X} \times \mathcal{A}$, the impact surface is $\mathcal{S}_{aux} := \mathcal{S} \times \mathcal{A}$, and the differential equation and impact map are given by

$$\begin{aligned} f_{aux}(x, a) &= \begin{bmatrix} f(x, a) \\ 0 \end{bmatrix} \\ \Delta_{aux}(x, a) &= \begin{bmatrix} \Delta(x, v_1(x)) \\ v_1(x) \end{bmatrix}. \end{aligned} \tag{19}$$

The hypotheses of Theorem 4 assure that (18) and $\mathbf{Z} := \{(Z_a, a) \mid a \in \mathcal{A}\}$ satisfy all the hypotheses of Theorem 2, and thus the existence and stability of orbits can be checked by evaluating the stability of fixed points of the discrete-time system associated with the restricted Poincaré map, namely

$$\begin{aligned} x[k+1] &= \rho(x[k], v_1(x[k])) \\ a[k+1] &= v_1(x[k]). \end{aligned} \tag{20}$$

Since the stability properties of (20) are equivalent to those of

$$x[k+1] = \rho(x[k], v_1(x[k])), \tag{21}$$

the result is proven.

For $\mathcal{W} \neq \emptyset$, the reasoning is essentially identical and is left to the reader. ■

We next allow $\mathcal{S} \cap Z_a$ to depend on $a \in \mathcal{A}$ and hence impact invariance must be replaced by a more general notion that is closer to what has been used in transition control [18].

Theorem 5: (Stability under Event-Based Parameter Updates-III) Consider the collection of systems with impulse effects, (8), with a taking values in $\mathcal{A} := \mathcal{A}_1 \times \mathcal{A}_2$, where $\mathcal{A}_1$ is an open subset of $\mathbb{R}^{p_1}$ and $\mathcal{A}_2$ is an open subset of $\mathbb{R}^{p_2}$. Suppose that $\mathcal{X}$ and $\mathcal{S}$ satisfy Hypotheses HSH0), HSH2)–HSH4). Suppose furthermore that Hypothesis HSH1) holds for the differential equation (9) and there exists a collection of subsets of $\mathcal{X}$ such that:

1) $\forall (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$, $Z_{a_1, a_2} \subset \mathcal{X}$ satisfies Hypotheses InvH1) and InvH2);
2) $\forall (a_1, a_2) \in \mathcal{A}_1 \times \mathcal{A}_2$, $\mathcal{S} \cap Z_{a_1, a_2}$ is independent of a_1; denote the intersection with $\mathcal{S}$ by $\mathcal{S} \cap Z_{\diamondsuit, a_2}$;
3) there exists a continuous function $\psi : \mathcal{A}_2 \to \mathcal{A}_1$ such that, $\forall a_2^-, a_2^+ \in \mathcal{A}_2$, $\Delta(\mathcal{S} \cap Z_{\diamondsuit, a_2^-}, \psi(a_2^-), a_2^+) \subset Z_{\psi(a_2^-), a_2^+}$.
4) $\mathbf{Z} := \{(x, a_1, a_2) \mid x \in Z_{a_1, a_2}, a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2\}$ is an embedded submanifold of $\mathcal{X} \times \mathcal{A}_1 \times \mathcal{A}_2$ and is locally continuously finite-time attractive for (9).

Let $\mathcal{W}$ be an open subset of $\mathbb{R}^\ell$ and define $\mathcal{X}_{aux} := \mathcal{X} \times \mathcal{A} \times \mathcal{W}$ and $\mathcal{S}_{aux} := \mathcal{S} \times \mathcal{A} \times \mathcal{W}$. Suppose that $v_1 : \mathcal{S} \times \mathcal{W} \to \mathcal{A}_2$ and $v_2 : \mathcal{S} \times \mathcal{W} \to \mathcal{W}$ are continuous. Define $\mathcal{X}_{aux} := \mathcal{X} \times \mathcal{A} \times \mathcal{W}$, $\mathcal{S}_{aux} := \mathcal{S} \times \mathcal{A} \times \mathcal{W}$, and $\mathcal{Z}_{aux} := \mathbf{Z} \times \mathcal{W}$. Then,

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{a}_1 \\ \dot{a}_2 \\ \dot{w} \end{bmatrix} &= \begin{bmatrix} f(x, a_1, a_2) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad & \begin{bmatrix} x^- \\ a_1^- \\ a_2^- \\ w^- \end{bmatrix} \notin \mathcal{S}_{aux} \\ \begin{bmatrix} x^+ \\ \\ a_1^+ \\ a_2^+ \\ w^+ \end{bmatrix} &= \begin{bmatrix} \Delta(x^-, \psi(a_2^-), \\ \quad v_1(x^-, w^-)) \\ \psi(a_2^-) \\ v_1(x^-, w^-) \\ v_2(x^-, w^-) \end{bmatrix}, \qquad & \begin{bmatrix} x^- \\ a_1^- \\ a_2^- \\ w^- \end{bmatrix} \in \mathcal{S}_{aux}, \end{aligned} \tag{22}$$

has a stable (resp., asymptotically stable) orbit transversal to $\mathcal{S}_{aux} \cap \mathcal{Z}_{aux}$ if, and only if, the discrete-time system

$$\begin{aligned} x[k+1] &= \rho(x[k], \psi(a_2[k]), v_1(x[k], w[k])) \\ a_2[k+1] &= v_1(x[k], w[k]) \\ w[k+1] &= v_2(x[k], w[k]) \end{aligned} \tag{23}$$

on $\{(\mathcal{S} \cap Z_{\diamondsuit,a_2}, a_2) \mid a_2 \in \mathcal{A}_2\} \times \mathcal{W}$ has a stable (resp., asymptotically stable) equilibrium point $(x^*; a_2^*; w^*)$ such that $L_f H(x^*, a_1^*, a_2^*) \neq 0$, where $a_1^* = \psi(a_2^*)$. ♦

Proof: The proof follows the same pattern as the proof of Theorem 4. For clarity, first assume that $\mathcal{W} = \emptyset$ and consider

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{a}_1 \\ \dot{a}_2 \end{bmatrix} &= f_{aux}(x, a_1, a_2) & \begin{bmatrix} x^- \\ a_1^- \\ a_2^- \end{bmatrix} &\notin \mathcal{S}_{aux} \\ \begin{bmatrix} x^+ \\ a_1^+ \\ a_2^+ \end{bmatrix} &= \Delta_{aux}(x^-, a_1^-, a_2^-) & \begin{bmatrix} x^- \\ a_1^- \\ a_2^- \end{bmatrix} &\in \mathcal{S}_{aux}, \end{aligned} \tag{24}$$

where the state space is $\mathcal{X}_{aux} := \mathcal{X} \times \mathcal{A}_1 \times \mathcal{A}_2$, the impact surface is $\mathcal{S}_{aux} := \mathcal{S} \times \mathcal{A}_1 \times \mathcal{A}_2$, and the differential equation and impact map are given by

$$\begin{aligned} f_{aux}(x, a_1, a_2) &= \begin{bmatrix} f(x, a_1, a_2) \\ 0 \\ 0 \end{bmatrix} \\ \Delta_{aux}(x, a_1, a_2) &= \begin{bmatrix} \Delta(x, \psi(a_2), v_1(x)) \\ \psi(a_2) \\ v_1(x) \end{bmatrix}. \end{aligned} \tag{25}$$

The hypotheses of Theorem 5 assure that (24) and $Z_{aux} := \mathbf{Z}$ satisfy all the hypotheses of Theorem 2 and thus the existence and stability of orbits can be checked by evaluating the stability of fixed points of the discrete-time system associated with the restricted Poincaré map, namely

$$\begin{aligned} x[k+1] &= \rho(x[k], \psi(a_2[k]), v_1(x[k])) \\ a_1[k+1] &= \psi(a_2[k]) \\ a_2[k+1] &= v_1(x[k]) \end{aligned} \tag{26}$$

Since the stability properties of (26) are equivalent to those of

$$\begin{aligned} x[k+1] &= \rho(x[k], \psi(a_2[k]), v_1(x[k])) \\ a_2[k+1] &= v_1(x[k]), \end{aligned} \tag{27}$$

the result is proven. The simple modifications for including $\mathcal{W} \neq \emptyset$ are left to the reader. ■

IV. Conclusion

This note has analyzed the problem of event-based feedback control of systems with impact effects, with the particular objective of creating and asymptotically stabilizing periodic orbits. The method of Poincaré sections transforms the analysis of periodic orbits into one of analyzing fixed points of the Poincaré return map, which in turn is equivalent to analyzing equilibrium points of a sampled-data system evolving on the impact surface. Parameter values that are held constant within the continuous phase of the hybrid dynamics and updated at impact events appear as standard sampled-data controls in the Poincaré return map. In many practical instances that have only been alluded to in this note, but which have been developed in detail elsewhere [18], [8], [17], [5], it is very advantageous to design the continuous-phase controller so that it creates a hybrid subsystem in the system with impulse effects. It was shown how this could be exploited in event-based control designs.

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