A LINEAR ALGEBRAIC FRAMEWORK FOR THE ANALYSIS OF DISCRETE-TIME NONLINEAR SYSTEMS*

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Abstract. A linear algebraic framework for the analysis of synthesis-type problems for discrete-time nonlinear systems is introduced. This is an extension of a similar tool for continuous-time systems that established important connections between many algorithms associated with right-invertibility, left-invertibility and dynamic decoupling, as well as between these algorithms and an approach based upon differential algebra. A similar payoff is seen to be possible in the discrete-time setting.

Key words. nonlinear systems, discrete-time, ranks, invertibility

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1. Introduction. This paper extends to the class of discrete-time nonlinear systems the linear algebraic framework of [4], which has proven useful in the analysis of several synthesis problems for the class of continuous-time nonlinear systems [1]–[3], [13], [16], [32]. Recall that [4], through the introduction of a chain of subspaces naturally associated with the output of a system, provided a high-level interpretation of the inversion and dynamic decoupling algorithms that are built around the recursive computation of certain ranks associated with left-invertibility, right-invertibility, and noninteracting control. In addition, it established relationships between these algorithms and the differential algebraic approach. This same linear algebraic setting has been used in [3] to formulate in an intrinsic way the regularity (constant rank) conditions common to several procedures for synthesizing nonlinear dynamic compensators.

The reader is reminded that the importance of algebraic techniques and reasoning for analyzing many aspects of discrete-time nonlinear systems has been firmly established in [8]–[12], [25], [28], [29], and the references therein. The techniques employed here are most closely related to those of [10] and [28].

When studying continuous-time nonlinear systems, the class of affine systems (so-called because the dynamics is affinely parametrized by the control variables) has received the bulk of the attention of the nonlinear community. This is true for several reasons, the most important of which is that the class of affine systems is general enough to encompass many models arising in practice. However, it is also specific enough to admit reasonably simple analyses from at least two perspectives: *geometrically*, we are working with a finite number of vector fields, a drift term, and m control vector fields, as opposed to some arbitrarily, smoothly parametrized family of vector fields; *algebraically*, the various derivatives of the outputs depend polynomially on the inputs and their derivatives (with coefficients depending on the state, and the highest order derivative of the input appearing affinely), as opposed to some more general nonlinear dependence on the input. However, if we accept as an axiom that

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any interesting class of discrete-time systems should include time-sampled (digital) versions of the class of continuous-time affine systems, then we are obliged to consider systems of the form

$$x[k + 1] = f(x[k], u[k]),$$
$$y[k] = h(x[k], u[k]),$$

where f and h are sufficiently smooth functions, but otherwise arbitrary (consider sampling a continuous-time bilinear system). Consequently, it is not possible to assume that the dynamics is affine in the control variable (and hence, finitely parametrized); and even if it were, this would not entail that the iterates of the output depend polynomially on the inputs, with the highest-order delayed input appearing affinely. Consequently, the proof techniques of [4], based upon "global" interpretations of the inversion and dynamic extension algorithms, cannot be easily extended to discrete-time systems; a more intrinsic, "algorithm-free" analysis will be performed.

In §2 of this paper, the linear algebraic framework of [4] is developed for analytic discrete-time systems, thereby extending the notion of the rank of a system introduced in [10]. This includes the definition of a chain of subspaces constructed from the outputs of the system and their iterates and an analysis of the convergence properties of the chain of subspaces. It is noted that when the function f describing the dynamics is not a submersion, certain new phenomena can occur, requiring a slightly different analysis involving a combination of geometric and algebraic reasoning. Section 3 collects a few results that are useful for establishing relations between some existing work involving rank computations in an "algorithmic" form and the linear algebraic setting proposed here. Section 4 relates the abstract notion of rank, introduced in §2, to the injectivity and surjectivity properties of certain maps strongly connected with left- and right-invertibility. Finally, §5 points out the links between the approach used in this paper and that of [10]; §6 shows the affinity with the work of [28].

2. Rank and structure at infinity. The notion of the *rank* of a nonlinear system was introduced by Fliess in [8] and yielded fundamental results on right- and left-invertibility and noninteracting control of continuous-time systems. Extensions to a class of rational discrete-time systems have been given in [10], using difference algebra in place of differential algebra. Here, using elementary vector space techniques as in [4], the rank of a discrete-time system will be generalized to analytic systems admitting a global state space representation on $I\!R^n$. This may be a strong assumption.

2.1. Linear algebraic framework. Consider a discrete-time system

(2.1) $$\Sigma : \begin{aligned} x[k+1] &= f(x[k], u[k]), \\ y[k] &= h(x[k], u[k]), \end{aligned}$$

where $x[k] \in X = I\!R^n$, $u[k] \in U = I\!R^m$, $y[k] \in Y = I\!R^\mu$, f and h are analytic functions of their arguments, and $x[0] = x_0$. It is convenient to let $f^u(x) := f(x, u)$ so that we may write:

$$x[1] = f^{u[0]}(x_0),$$
$$x[2] = f^{u[1]}(x[1]) = f^{u[1]} \circ f^{u[0]}(x_0)$$
$$\vdots$$
$$x[k] = f^{u[k-1]} \circ \cdots \circ f^{u[0]}(x_0),$$

where ∘ denotes composition. Then, since $y[k] = h^{u[k]}(x[k])$, where $h^u(x) := h(x, u)$,

$$
\begin{aligned}
y[k] &= y[k] \, (x_0, u[0], \ldots, u[k]) \\
&= h^{u[k]} \circ f^{u[k-1]} \circ \cdots \circ f^{u[0]}(x_0).
\end{aligned}
\tag{2.2}
$$

Because (2.1) is time-invariant,

$$
\begin{aligned}
y[k+1] &= y[k] \, (x[1], u[1], \ldots, u[k+1]) \\
&= y[k] \, (f \, (x_0, u[0]) \,, u[1], \ldots, u[k+1]),
\end{aligned}
\tag{2.3}
$$

which will be important later when establishing a certain finiteness property.

Let $\mathcal{R}_k$ denote the ring of real analytic functions of the components of $(x, u[0], \ldots, u[k])$, and let $\mathcal{K}_k$ be the associated field of fractions, that is, the field of *meromorphic* functions in the variables $(x, u[0], \ldots, u[k])$. A typical element of $\mathcal{K}_k$ would have the form $\eta(v) = \pi(v)/\theta(v)$, where π and θ are elements of $\mathcal{R}_k$, θ is not the zero function, and $v = (v_1, \ldots, v_j)$ denotes the various components of $(x, u[0], \ldots, u[k])$. Recall that $\partial/\partial v_i$ acting on η is formally defined by the usual quotient rule of calculus,

$$
\frac{\partial}{\partial v_i} \frac{\pi(v)}{\theta(v)} := \left(\theta(v) \frac{\partial}{\partial v_i} \pi(v) - \pi(v) \frac{\partial}{\partial v_i} \theta(v) \right) / \theta^2(v),
\tag{2.4}
$$

and the formal differential of η is

$$
d\eta(v) := \sum_{i=1}^{j} \frac{\partial \eta(v)}{\partial v_i} dv_i.
\tag{2.5}
$$

Let $\mathcal{E}$ denote the vector space over $\mathcal{K} := \mathcal{K}_n$ spanned by $\{dx_1, \ldots, dx_n, du_1[0], \ldots, du_m[0], du_1[n], \ldots, du_m[n]\}$. Note that $\mathcal{E}$ is a finite-dimensional vector space; indeed, its dimension is $n + (n+1)m$. For notational convenience, $\{dx_1, \ldots, dx_n\}$ will simply be written as dx, $\{du_1[0], \ldots, du_m[0]\}$ as $du[0]$, etc., so that $\mathcal{E} = \text{span}_{\mathcal{K}}\{dx, du[0], \ldots, du[n]\}$.

Observe now, for all $0 \le k \le n$ and $1 \le j \le \mu$, that $dy_j[k] \in \mathcal{E}$, since

$$
dy_j[k] = \sum_{i=1}^{n} \frac{\partial y_j[k]}{\partial x_i} dx_i + \sum_{\ell=0}^{k} \sum_{i=1}^{m} \frac{\partial y_j[k]}{\partial u_i[\ell]} du_i[\ell].
\tag{2.6}
$$

Define a chain of subspaces $\mathcal{E}_0 \subset \cdots \subset \mathcal{E}_n$ of $\mathcal{E}$ by [4] (see also [7])

$$
\mathcal{E}_k := \text{span}_{\mathcal{K}}\{dx, dy[0], \ldots, dy[k]\}
\tag{2.7}
$$

and the associated list of dimensions $\rho_0 \le \rho_1 \le \cdots \le \rho_n$ by

$$
\rho_k = \dim_{\mathcal{K}} \mathcal{E}_k.
\tag{2.8}
$$

We emphasize that $dy[k]$ denotes $\{dy_1[k], \ldots, dy_\mu[k]\}$ and that this abuse of notation will be used quite often to keep the notation compact.

It will turn out for *generically submersive* systems,[1] that is, for systems where

$$
n = \text{rank}_{\mathcal{K}} \left[\frac{\partial f}{\partial x}(x, u[0]) \vdots \frac{\partial f}{\partial u}(x, u[0]) \right],
\tag{2.9}
$$

---

[1] The mathematical importance of this assumption will be seen in the next subsection; in terms of control systems, it means that with a feedback, the drift dynamics could be made (generically) invertible, creating a kind of group action.

that

$$(2.10) \qquad \rho^* := \rho_n - \rho_{n-1}$$

is a limiting value of the chain (2.7) in the sense that if we were to extend the chain in the obvious manner, then $\rho_{n+r} = \rho_n + r\rho^*$, for all integers $r \geq 0$. Many system models of the form (2.1) would satisfy (2.9), since it is equivalent to $f(I\!\!R^n, I\!\!R^m)$ having nonempty interior in $I\!\!R^n$, and this is a *necessary* condition for accessibility [17]. It is always satisfied for a time-sampled representation of a continuous-time system. Moreover, it has just been established in [11] that, in a certain sense, rational input-output systems admit *local* state space representations satisfying condition (2.9). To avoid passing to a local representation, the following construction is used here in the general case where (2.9) is not satisfied.

Let $\mathcal{K}^+ := \mathcal{K}_{2n}$, and define $\mathcal{E}^+ := \mathrm{span}_{\mathcal{K}^+}\{dx, du[0], \ldots, du[2n]\}$. Define a chain of subspaces $\mathcal{E}_0^+ \subset \cdots \subset \mathcal{E}_n^+$ of $\mathcal{E}^+$ by

$$(2.11) \qquad \mathcal{E}_k^+ := \mathrm{span}_{\mathcal{K}^+}\{dx, du[0], \ldots, du[n-1], dy[n], \ldots, dy[n+k]\}$$

and the associated list of dimensions $\rho_0^+ \leq \cdots \leq \rho_n^+$ by

$$(2.12) \qquad \rho_k^+ := \dim \mathcal{E}_k^+.$$

Then, even without condition (2.9), it will turn out that

$$(2.13) \qquad \rho^{+*} := \rho_n^+ - \rho_{n-1}^+$$

is a limiting value in the sense discussed earlier for ρ^*. Whenever the system (2.1) is generically submersive, it will be established that

$$(2.14) \qquad \rho_k^+ = \rho_k + nm, \qquad k \geq 0,$$

so that $\rho^{+*} = \rho^*$.

Anticipating these technical results, ρ^{+*} is defined to be the *rank* [10] of the system (2.1).

Remarks. (a) In [6], it is shown (for continuous-time systems) that the chain of subspaces (2.7) is closely related to classical objects in algebra, namely filtrations, and consequently, Hilbert polynomials; a similar result is true in discrete-time [7]. One of the main points of the analysis presented in this paper is the establishment of a priori bounds on the number of steps required to compute the limiting ranks of the filtrations whenever the system has a standard state-space representation; such bounds are not provided by the classical results of algebra, which, on the other hand, apply to more general situations.

(b) In analogy with [4], [22], the list of integers $\{\sigma_0, \ldots, \sigma_n\}$ defined by

$$(2.15) \qquad \sigma_i = \rho_i - \rho_{i-1}, \quad 0 \leq i \leq n$$

with the convention that $\rho_{-1} = n$, could be called the *transient structure at infinity*, while the list of integers $\{\sigma_0^+, \ldots, \sigma_n^+\}$, defined by

$$(2.16) \qquad \sigma_i^+ := \rho_i^+ - \rho_{i-1}^+, \quad 0 \leq i \leq n,$$

with the convention that $\rho_{-1}^+ = n + nm$ could be called the *persistent structure at infinity*. For generically submersive systems the two lists coincide, and we can speak simply of the *structure at infinity*; this is also the case for systems satisfying certain constant rank hypotheses in the neighborhood of an equilibrium point, as can be seen from the results of [18], [24], including (constant-coefficient) linear systems.

2.2. Convergence of the chain $\mathcal{E}_0 \subset \cdots \subset \mathcal{E}_n$. The goal of this subsection is to justify terminating the chain (2.7) at n, the dimension of the state space of (2.1), whenever the system is generically submersive. For a linear system, this would follow as an easy consequence of the Cayley–Hamilton theorem; in the case of nonlinear systems, more work is required.

Let $k \geq 0$ be any nonnegative finite integer, and recall that $\mathcal{K}_k$ is the field of meromorphic functions of $(x, u[0], \ldots, u[k])$. Before, when defining $\mathcal{E}_k$ ($0 \leq k \leq n$), the span was taken with respect to $\mathcal{K} := \mathcal{K}_n$. It is easily seen that the dimension of $\mathcal{E}_k$ does not change if instead the span is taken with respect to $\mathcal{K}_k$. For $k > n$, define $\mathcal{E}_k$ in the obvious way, following (2.7), taking the span with respect to $\mathcal{K}_k$.

THEOREM 2.1. *Suppose that (2.1) is generically submersive. Then, for all integers $k \geq n$, $\dim \mathcal{E}_k - \dim \mathcal{E}_{k-1} = \dim \mathcal{E}_n - \dim \mathcal{E}_{n-1}$; that is, $\rho_k - \rho_{k-1} = \rho_n - \rho_{n-1}$.*

The proof of the theorem will be divided into several parts, each establishing a particular property of the chain (2.7) arising from the recursive manner in which the functions $y[k]$ are constructed from the system. Define $\delta : \mathcal{K}_{k-1} \to \mathcal{K}_k$ by

$$(2.17) \qquad (\delta\eta)(x, u[0], \ldots, u[k]) = \eta(f(x, u[0]), u[1], \ldots, u[k]);$$

it is important that f be generically submersive, for otherwise, $\delta\eta$ may not be a meromorphic function (see (2.20) below). This induces an $\mathbb{R}$-linear mapping[2] Δ: $\text{span}_{\mathcal{K}_{k-1}}\{d\eta | \eta \in \mathcal{K}_{k-1}\} \to \text{span}_{\mathcal{K}_k}\{d\lambda | \lambda \in \mathcal{K}_k\}$ by

$$(2.18) \qquad \begin{aligned} &\Delta(d\eta) := d(\delta\eta), \\ &\Delta(\alpha_1 d\eta_1 + \alpha_2 d\eta_2) := \delta(\alpha_1)\Delta(d\eta_1) + \delta(\alpha_2)\Delta(d\eta_2) \end{aligned}$$

for $\eta, \eta_1, \eta_2, \alpha_1, \alpha_2 \in \mathcal{K}_{k-1}$. It should be noted that (2.18) is consistent with the chain rule for differentiation, and that, for instance,

$$(2.19) \qquad \begin{aligned} dy_j[k+1] &= d\left(\delta(y_j[k])\right) \\ &= \Delta\left(dy_j[k]\right) \\ &= \sum_{i=1}^{n} \delta\left(\frac{\partial y_j[k]}{\partial x_i}\right) dx_i[1] + \sum_{\ell=0}^{k}\sum_{i=1}^{m} \delta\left(\frac{\partial y_j[k]}{\partial u_i[\ell]}\right) du_i[\ell+1]. \end{aligned}$$

The following two (equivalent) properties are easily established whenever the system (2.1) is generically submersive:

P1. For all $k \geq 1$, $\dim_{\mathcal{K}_{k-1}}\{dx[k]\} = n$.

P2. For all $k \geq 0$ and $\forall \eta \in \mathcal{R}_k$, if $\eta \not\equiv 0$, then $\delta(\eta) \not\equiv 0$.

As a consequence, if $\eta \in \mathcal{K}_k$, then $\delta(\eta)$ is well defined. To see the peculiarities a nonsubmersive system may exhibit, consider the example

$$(2.20) \qquad \begin{aligned} x[k+1] &= 0, \\ y[k] &= x[k]u[k], \end{aligned}$$

where x, y, and u are in $\mathbb{R}$. The function $1/x$ is meromorphic, but $\delta(1/x)$ is not defined because $\delta(x) \equiv 0$.

The main ingredients of a proof of Theorem 2.1 are now presented. In the following, if S is a set, then $|S|$ denotes its cardinality.

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[2] The fact that Δ is well defined follows easily from $\Delta(0) = 0$.

LEMMA 2.2. *Suppose that (2.1) is generically submersive and that for some $k \geq 0$,
$I_0 \subset I_1 \subset \cdots \subset I_k \subset \{1, \ldots, \mu\}$ are index sets such that $\mathcal{E}_k = \operatorname{span}_{\mathcal{K}_k}\{dx, dy_{i_0}[0], \ldots,
dy_{i_k}[k]|i_j \in I_j, \ 0 \leq j \leq k\}$, and $\dim \mathcal{E}_k = n + |I_0| + \ldots + |I_{k-1}| + |I_k|$. Then,
$\dim \operatorname{span}_{\mathcal{K}_k}\{dx, dy_{i_0}[0], \ldots, dy_{i_k}[k], dy_{i_k}[k+1]|i_j \in I_j, \ 0 \leq j \leq k\} = n + |I_0| + \ldots +
|I_{k-1}| + 2|I_k|$. In other words, once an output component becomes independent, it
remains independent.*

Proof. For the proof, see Appendix A. □

It immediately follows from the above that there exists a basis of a special form
for the chain $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots$.

LEMMA 2.3. *Suppose that (2.1) is generically submersive. Then, there exist index
sets $I_0 \subset I_1 \subset \cdots \subset \{1, \ldots, \mu\}$ such that, for all $k \geq 1$, $\{dx, dy_{i_0}[0], \ldots, dy_{i_k}[k]|i_j \in
I_j, \ 0 \leq j \leq k\}$ is a basis for $\mathcal{E}_k$.*

Lemma 2.3 establishes that $\{\sigma_k\} = \{\rho_k - \rho_{k-1}\}$ is a nondecreasing sequence. Since
$\sigma_k \leq \min\{m, \mu\}$, it follows that $\{\sigma_k\}$ converges in a finite number of steps. However,
this does not allow us to terminate the calculations at the nth step unless the upper
bound has already been attained. Considering once again (2.20), we calculate that
$(\sigma_0) = 1$, but $(\sigma_1) = 0$; that is, the sequence $\{\sigma_k\}$ of "zeros at infinity of order less
than or equal to k" is not nondecreasing, as is always the case for linear systems and
continuous-time nonlinear systems.

LEMMA 2.4. *Suppose that (2.1) is generically submersive, and let $I_0, \ldots, I_n$ be as
in Lemma 2.3. Then, for each $1 \leq j \leq \mu$, there exists an integer N, $1 \leq N \leq n$, such
that*

(2.21)

$$dy_j[N] \in \operatorname{span}_{\mathcal{K}_N}\{dy_j[0], \ldots, dy_j[N-1], dy_{i_0}[0], \ldots, dy_{i_N}[N]|i_k \in I_k, \ 0 \leq k \leq N\}.$$

Proof. For the proof, see Appendix A. □

The previous and the following lemmas combine to replace the Cayley–Hamilton
theorem, which, in the case of a linear system, proves that the chain (2.7) converges
in at most n steps.

LEMMA 2.5. *Suppose that (2.1) is generically submersive, and let $I_0, \ldots, I_n$ be as
in Lemma 2.3. Suppose that $1 \leq j \leq \mu$, and let N be as in Lemma 2.4. Then, for all
$k \geq N$,*

(2.22)

$$dy_j[k] \in \operatorname{span}_{\mathcal{K}_k}\{dy_j[0], \ldots, dy_j[N-1], dy_{i_0}[0], \ldots, dy_{i_k}[k]|$$
$$\text{for } 1 \leq s \leq N, i_s \in I_s, \text{ and for } s > N, i_s \in I_N\}.$$

Proof. The proof is immediate from (2.18) and Lemma 2.4. □

The proof of Theorem 2.1 is now given easily. Let $\{I_k\}$ be the collection of index
sets determined by Lemma 2.2. By Lemma 2.5, $I_{n+r} = I_n$ for all $r \geq 1$. Hence,
the components of the output either become independent by the nth iteration of the
dynamics, or they remain dependent for all iterations. Consequently, for all $r \geq 1$,
$\dim \mathcal{E}_{n+r} - \dim \mathcal{E}_{n+r-1} = \dim \mathcal{E}_n - \dim \mathcal{E}_{n-1} = \rho^*$.

2.3. Convergence of the chain $\mathcal{E}_0^+ \subset \mathcal{E}_1^+ \subset \cdots$. This section addresses the
convergence properties of the chain $\mathcal{E}_0^+ \subset \mathcal{E}_1^+ \subset \cdots$ for systems that are not necessarily
submersive. The idea behind the analysis is that the effect of the nilpotent part of
the system on the output sequence is short lived and can be eliminated from the
analysis by "ignoring" the first n time instances of the output; this is essentially what

is accomplished by including $\{du[0], \ldots, du[n-1]\}$ in the definition of $\mathcal{E}_0^+ \subset \mathcal{E}_1^+ \subset \cdots$.
The analytical aspects of the proof are based on the following construction.

Let $M_0 := \mathbb{R}^n$ and define for $k \geq 1$

$$(2.23) \qquad\qquad M_k := x[k] \left(\mathbb{R}^n \times (\mathbb{R}^m)^k \right).$$

Observe that $M_0 \supset M_1 \supset \cdots$ since

$$(2.24) \qquad\qquad M_{k+1} = f(M_k, \mathbb{R}^m).$$

Define $d_0 := n$ and for $k \geq 1$,

$$(2.25) \qquad\qquad d_k := \mathrm{rank}_{\mathcal{K}_{k-1}} x[k] := \dim \mathrm{span}_{\mathcal{K}_{k-1}} \{dx[k]\}.$$

LEMMA 2.6. *The sequence of integers $\{d_k\}$ is nonincreasing, and if $d_k = d_{k-1}$
then $d_{k+1} = d_k$. Consequently, since d_j can decrease at most n-times, $d_j = d_n$ for all
$j \geq n$.*

The proof is given in Appendix B. Very roughly speaking, Lemma 2.6 says that
"$f : M_k \times \mathbb{R}^m \to M_{k+1}$ behaves like a 'submersion' for $k \geq n$ since the 'dimension' of
M_k is d_k." Of course, M_k, in general, does not have the structure of a manifold, hence,
the imprecision of such a statement. Nevertheless, if we pursue this line of thought
for a moment and supposes that $M_0 \supset M_1 \supset \cdots$ is a nested sequence of embedded
analytic submanifolds, it is clear that, if $\dim M_k = \dim M_{k-1}$, then $\dim M_{k+1} =
\dim M_k$, because the condition $\dim M_k = \dim M_{k-1}$ implies that $f : M_{k-1} \times \mathbb{R}^m \to
M_k$ is a submersion. This combined with M_k being open in M_{k-1} gives the result.
The proof in Appendix B makes this line of reasoning rigorous with a local analysis,
which also establishes the following result: For $k \geq 2$, $j \geq 0$ define $\delta^k : \mathcal{R}_j \to \mathcal{R}_{j+k}$
by, if $\alpha \in \mathcal{R}_j$, and $0 \leq i \leq k-1$

$$(2.26) \qquad\qquad \delta^{i+1}(\alpha) = \delta(\delta^i(\alpha)).$$

LEMMA 2.7. *Let $k \geq 0$, $\alpha \in \mathcal{R}_k$. If $\delta^n(\alpha) \not\equiv 0$ (i.e., is not the identically zero
function), then $\delta^{n+j}(\alpha) \not\equiv 0$ for all $j \geq 1$.*

Let $\mathcal{R}_k^{\delta^n}$ be the set of real analytic functions

$$(2.27) \qquad\qquad \mathcal{R}_k^{\delta^n} := \{\delta^n(\alpha) | \alpha \in \mathcal{R}_k\}.$$

$\mathcal{R}_k^{\delta^n} \subset \mathcal{R}_{n+k}$ as a subring, and thus, the associated set of fractions, denoted $\mathcal{K}_k^{\delta^n}$, is a
field; indeed, it is a subfield of $\mathcal{K}_{n+k}$. Note that $\mathcal{K}_k^{\delta^{n+1}} \subset \mathcal{K}_{k+1}^{\delta^n}$.

By Lemma 2.7, for each $k \geq 0$, the two mappings

$$(2.28) \qquad\qquad \delta : \mathcal{K}_k^{\delta^n} \to \mathcal{K}_k^{\delta^{n+1}} \subset \mathcal{K}_{k+1}^{\delta^n}$$

and

(2.29)

$$\Delta : \mathrm{span}_{\mathcal{K}_k^{\delta^n}} \{d\lambda | \lambda \in \mathcal{K}_k^{\delta^n}\} \to \mathrm{span}_{\mathcal{K}_k^{\delta^{n+1}}} \{d\eta | \eta \in \mathcal{K}_k^{\delta^{n+1}}\} \subset \mathrm{span}_{\mathcal{K}_{k+1}^{\delta^n}} \{d\gamma | \gamma \in \mathcal{K}_{k+1}^{\delta^n}\}$$

can be defined as in (2.17) and (2.18), respectively.

The final step of the analysis is to reduce the study of the chain $\mathcal{E}_0^+ \subset \mathcal{E}_1^+ \subset \cdots$ to
that of a related chain to which the proof technique of §2.2, that is, Lemmas 2.2–2.5,

can be applied with only minor modifications. As in §2.2, it is necessary to slightly modify the definition of $\mathcal{E}_k^+$ without changing its dimension:

$$(2.30) \qquad \mathcal{E}_k^+ := \mathrm{span}_{\mathcal{K}_{n+k}}\{dx, du[0], \ldots, du[n-1], dy[n], \ldots, dy[n+k]\},$$

for all $k \geq 0$, and

$$(2.31) \qquad \mathcal{E}_{-1}^+ := \mathrm{span}_{\mathcal{K}_{n-1}}\{dx, du[0], \ldots, du[n-1]\}.$$

Note that $\{dx[n]\} \subset \mathcal{E}_{-1}^+$. Let $L \subset \{1, \ldots, n\}$ be such that $\{dx_i[n]|i \in L\}$ is a basis for $\mathrm{span}_{\mathcal{K}_{n-1}}\{dx[n]\}$, and let $\mathcal{W}$ be such that

$$(2.32) \qquad \mathcal{E}_{-1}^+ = \mathrm{span}\{dx_i[n]|i \in L\} \oplus \mathcal{W}.$$

Introduce

$$\tilde{\mathcal{S}}_k := \mathrm{span}_{\mathcal{K}_{n+k}}\{dx_i[n], dy[n], \ldots, dy[n+k]|i \in L\},$$
$$(2.33) \qquad \tilde{\mathcal{S}}_{-1} := \mathrm{span}_{\mathcal{K}_{n-1}}\{dx_i[n]|i \in L\}.$$

Since $y[n+j](x, u[0], \ldots, u[n+j]) = y[j](x[n], u[n], \ldots u[n+j])$ for $j \geq 0$, it follows that

$$(2.34) \qquad \mathcal{E}_k^+ = \tilde{\mathcal{S}}_k \oplus \mathcal{W}, \qquad k \geq 1.$$

Hence, for $k \geq 0$,

$$(2.35) \qquad \sigma_k^+ = \dim\tilde{\mathcal{S}}_k - \dim\tilde{\mathcal{S}}_{k-1} .$$

The reason for doing all of this is that the generators of $\tilde{\mathcal{S}}_k$, for $k \geq 1$, are elements of $\{d\lambda|\lambda \in \mathcal{K}_k^{\delta^n}\}$. Hence, letting

$$\mathcal{S}_k := \mathrm{span}_{\mathcal{K}_k^{\delta^n}}\{dx_i[n], dy[n], \ldots, dy[n+k]|i \in L\},$$
$$(2.36) \qquad \mathcal{S}_{-1} := \mathrm{span}_{\mathcal{K}_{n-1}}\{dx_i[n]|i \in L\},$$

it follows that, for $k \geq 0$,

$$(2.37) \qquad \sigma_k^+ = \dim\mathcal{S}_k - \dim\mathcal{S}_{k-1}.$$

Moreover, even though $\mathcal{S}_k \subset \mathcal{S}_{k+1}$ is *not* a subspace of $\mathcal{S}_{k+1}$,

$$(2.38) \qquad \dim\ \mathrm{span}_{\mathcal{K}_{k+1}^{\delta^n}}\{\mathcal{S}_k\} = \dim\mathcal{S}_k,$$

and therefore, the proofs of the obvious modifications of Lemmas 2.2–2.5 for the chain $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots$ go through with only minor changes, which will not be repeated here.

THEOREM 2.8. *For all integers $k \geq n$, $\dim\mathcal{E}_k^+ - \dim\mathcal{E}_{k-1}^+ = \dim\mathcal{E}_n^+ - \dim\mathcal{E}_{n-1}^+$; that is, $\rho_k^+ - \rho_{k-1}^+ = \rho_n^+ - \rho_{n-1}^+$. Moreover, whenever (2.1) is generically submersive, the ordered lists $\{\sigma_0, \sigma_1, \ldots, \sigma_n\}$ and $\{\sigma_0^+, \ldots, \sigma_n^+\}$ are equal.*

The last part of the theorem follows from the fact that when (2.1) is generically submersive, the index set L in (2.32) is equal to $\{1, \ldots, n\}$. Then, since $y[n+j](x, u[0], \ldots, u[n+j])$ can be expressed as $y[j](x[n], u[n], \ldots, u[n+j])$, $\mathcal{E}_k$ and $\mathcal{S}_k$ are naturally isomorphic under $x \to x[n], u[0] \to u[n], \ldots, u[k] \to u[n+k]$.

3. Further characterizations of the rank and structure at infinity. This section and the rest of the paper will concentrate on generically submersive systems. Similar results, as per the development of §2.3, can be stated for the general case.

3.1. Jacobian matrices. The goal here is to provide a computationally convenient means of evaluating the rank ρ^*. The same result is also useful for showing the invariance of ρ^* under the action of invertible (static or dynamic) state variable feedback.

Following [14], which, in turn, was based upon [23], consider the Jacobian matrices

$$(3.1) \qquad J_k\left(x, u[0], \ldots, u[k]\right) := \frac{\partial(y[0], \ldots, y[k])}{\partial(u[0], \ldots, u[k])} \,,$$

for $0 \le k \le n$, and their associated ranks

$$(3.2) \qquad\qquad R_k := \mathrm{rank}_{\mathcal{K}} J_k.$$

Note that the matrices J_k can be evaluated symbolically; their ranks can be evaluated numerically since the rank over $\mathcal{K}$ is the same as the generic rank considered in [23].

Applying arguments identical to those used in [4, §2.1] results in the following relation between the integers ρ_k and R_k.

PROPOSITION 3.1. *For each* $0 \le k \le n$, $\rho_k = n + R_k$. *Hence, if* (2.1) *is generically submersive, then* $\rho^* = R_n - R_{n-1}$.

A quite different way of obtaining a result similar to the first part of Proposition 3.1 is given in [6].

Consider now a discrete-time linear system

$$(3.3) \qquad\qquad \begin{aligned} x[k+1] &= Ax[k] + Bu[k], \\ y[k] &= Cx[k] + Du[k]. \end{aligned}$$

Then the Jacobian matrix J_k is given by the usual Toeplitz matrix

$$(3.4) \qquad J_k = \begin{bmatrix} D & 0 & 0 \\ CB & D & \\ CAB & CB & \vdots \\ \vdots & & 0 \\ CA^{k-1}B & CA^{k-2}B & \cdots & D \end{bmatrix}.$$

The results of [21] and [26] in conjunction with Proposition 3.1 justify the terminology adopted in §2.2 concerning the rank and structure at infinity of a nonlinear system.

The following is the analogue of [10, III.B.2. Proposition].

COROLLARY 3.2. *In the case of a linear system, the rank* ρ^* *defined by* (2.10) *agrees with the classical rank of the transfer matrix. Moreover, the list of integers* $\{\sigma_0, \ldots, \sigma_n\}$ *defined in* (2.15) *is precisely the structure at infinity as it is normally defined on the basis of the transfer matrix* [21], [26].

REMARK 3.3. For a linear system, it is easy to verify that the lists $\{\sigma_k^+\}$ and $\{\sigma_k\}$ coincide, whether or not the system is generically submersive.

3.2. A related chain of subspaces. Related to the chain $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots$ is the chain $\mathcal{H}_0 \subset \mathcal{H}_1 \subset \cdots$ defined solely in terms of the output [4]:

$$(3.5) \qquad\qquad \mathcal{H}_k = \mathrm{span}_{\mathcal{K}_k}\{dy[0], \ldots, dy[k]\} \,.$$

It also can be used to determine the rank of the system, and this will be important for making contact with the fundamental work of [10].

THEOREM 3.4. *Suppose that (2.1) is generically submersive. For all integers* $k \geq n$, $\rho^* = \dim \mathcal{H}_k - \dim \mathcal{H}_{k-1}$.

Proof. By Lemma 2.5, for $k \geq n$ $\rho^* \geq \dim \mathcal{H}_k - \dim \mathcal{H}_{k-1}$. On the other hand, for $1 \leq j$, $\mathcal{H}_j = \mathcal{H}_{j-1} + \operatorname{span}\{dy[j]\}$, $\mathcal{E}_j = \mathcal{E}_{j-1} + \operatorname{span}\{dy[j]\}$ and $H_0 \subset \mathcal{E}_0$. Thus, $\dim \mathcal{H}_j - \dim \mathcal{H}_{j-1} \geq \dim \mathcal{E}_j - \dim \mathcal{E}_{j-1}$.   ☐

3.3. Remarks on the inversion algorithm. The importance of the inversion algorithm of Singh [27], which is an extension to nonlinear continuous-time systems of the well-known algorithm of Silverman [26], need not be underlined here. The algorithm has also been used in the study of discrete-time nonlinear systems [18] and [19], but always expressed in a form involving the implicit function theorem. Consequently, the results of the algorithm can be difficult to interpret unless one remains in a neighborhood of an equilibrium point. This problem can be removed by working at the level of the differentials of the outputs, which linearizes the computations and allows the analysis of [4] to be carried through to the discrete-time setting. Since the algorithm in the form we will use it has already appeared in several publications for continuous-time systems [3], [4], [16], the basic idea will only be sketched here by giving the first steps of the algorithm. Establishing the validity and convergence properties of the algorithm is quite easy using the analysis of §2.2.

It is assumed that (2.1) is generically submersive; an extension to general systems can be envisioned along the lines of §2.3.

Step 0. Calculate $dy[0]$ and write it as

$$(3.6) \qquad dy[0] = a_0(x, u[0])dx + b_0(x, u[0])du[0] \ .$$

Define

$$(3.7) \qquad s_0 := \operatorname{rank}_{\mathcal{K}_0} b_0 \ .$$

Permute, if necessary, the components of y so that the first s_0 rows of b_0 are linearly independent. Decompose y so that

$$(3.8) \qquad dy[0] = \begin{bmatrix} d\tilde{y}_0[0] \\ d\hat{y}_0[0] \end{bmatrix} = \begin{bmatrix} \tilde{a}_0 \\ \hat{a}_0 \end{bmatrix} dx + \begin{bmatrix} \tilde{b}_0 \\ \hat{b}_0 \end{bmatrix} du[0],$$

where $\tilde{y}_0$ has s_0 rows. Since the rows of $\hat{b}_0$ are $\mathcal{K}_0$-dependent on the rows of $\tilde{b}_0$, there exists a matrix $M_0(x, u[0])$ with entries in $\mathcal{K}_0$ such that

$$(3.9) \qquad \hat{b}_0 = M\tilde{b}_0,$$

and thus,

$$(3.10) \qquad \begin{aligned} d\hat{y}_0[0] &= \hat{a}_0 dx + M_0 \{ d\tilde{y}_0[0] - \tilde{a}_0 dx \} \\ &=: \bar{a}_0 dx + \bar{b}_0 d\tilde{y}_0[0]. \end{aligned}$$

End of Step 0.

Step 1. Compute

$$(3.11) \quad \begin{aligned} d\hat{y}_0[1] &= \left(\delta \bar{a}_0 \right) dx[1] + \left(\delta \bar{b}_0 \right) d\tilde{y}_0[1] \\ &=: a_1 \left(x, u[0], u[1] \right) dx + b_1 \left(x, u[0], u[1] \right) du[0] + c_1 \left(x, u[0], u[1] \right) d\tilde{y}_0[1]. \end{aligned}$$

Define

(3.12)
$$s_1 := \text{rank}_{\mathcal{K}_1} \begin{bmatrix} \tilde{b}_0 \\ b_1 \end{bmatrix}$$

and repeat the basic operations of Step 1; see [3], [4], and [16], for example.

The validation of the steps in the algorithm is achieved by noting that it produces a basis for $\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots$, and thus, by Lemma 2.2, since $\{dx, d\tilde{y}_0[0]\}$ is a linearly independent set, so is $\{dx, d\tilde{y}_0[0], d\tilde{y}_0[1]\}$, etc. Its convergence in no more than n steps follows from Theorem 2.1.

A similar connection with the interesting work of [24] on dynamic feedback solutions to the noninteracting control problem could be pursued also along the lines already clearly established in [4].

4. Invertibility. A linear system is usually said to be right-invertible if the rank of its transfer matrix is equal to the number of output components, and left-invertible if its rank equals the number of input components. Systemically, right-invertibility means that by a proper choice of the initial condition *and* input sequence, any output sequence can be generated; that is, the map from initial conditions and inputs is onto $Y^\infty = Y \times Y \times \cdots$, the space of all output sequences. Left-invertibility is equivalent to injectivity of the map from inputs to outputs, for a fixed initial condition.

In the case of nonlinear systems, though such global notions of invertibility are attractive, simple examples show the difficulty of trying to say anything intelligent about them; hence, we are led to localizing the concepts. Following [3], for $k \geq 0$, let $H_k : X \times U^{k+1} \to Y^{k+1}$ be the map that sends $(x, u[0], \ldots, u[k])$ to $(y[0], \ldots, y[k])$, and let $E_k : X \times U^{k+1} \to X \times Y^{k+1}$ by $(x, u[0], \ldots, u[k]) \longmapsto (x, y[0], \ldots, y[k])$.

DEFINITION 4.1. *The system* (2.1) *is almost everywhere locally surjective,* if, for every $k \geq 0$, the image of H_k has nonempty interior. *The system is almost everywhere locally injective,* if, for every $k \geq 0$, there exists an open and dense subset $\mathcal{O}_k$ of $X \times U^{k+1}$ with the property that, for each point $p = (x, u[0], \ldots, u[k]) \in \mathcal{O}_k$ there exists an open neighborhood of p, $\tilde{\mathcal{O}}_k(p)$, and an analytic insertion[3] $i_k : \tilde{\mathcal{O}}_k(p) \to X \times U^{k+1} \times U^n$ such that if $p_1, p_2 \in \tilde{\mathcal{O}}_k(p)$ and $E_{n+k}(i_k(p_1)) = E_{n+k}(i_k(p_2))$, then $p_1 = p_2$.

These properties can be characterized as follows.

THEOREM 4.2. *Assume that the nonlinear system* (2.1) *is generically submersive. Then the system is almost everywhere locally surjective if and only if any one of the following equivalent conditions is satisfied:*

(a) *for all* $k \geq 0, \dim \text{span}\{dy[0], \ldots, dy[k]\} = (k+1)\mu$;

(b) $\dim \text{span}\{dy[0], \ldots, dy[n]\} = (n+1)\mu$;

(c) $\rho^* = \mu$; *i.e., the rank of the system equals the number of output components. The system is almost everywhere locally injective if, and only if, any one of the following equivalent conditions is satisfied:*

(d) *for all* $k \geq 0$, $\{du[0], \ldots, du[k]\} \subset \text{span}\{dx, dy[0], \ldots, dy[k+n]\}$

(e) $du[0] \subset span\{dx, dy[0], \ldots, dy[n]\}$;

(f) $\rho^* = m$; *i.e., the rank of the system equals the number of input components.*

Proof. (a) $\Longrightarrow$ (b) is immediate. (b) $\Longrightarrow$ (c) is given by Theorem 3.4. (c) $\Longrightarrow$ (a) follows from the same kind of reasoning employed in proving Lemmas 2.4 and 2.5 and is not repeated here. It suffices to show that (a) is equivalent to almost everywhere

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[3] That is, if τ_k represents that natural projection of $X \times U^{k+1} \times U^n$ onto $X \times U^{k+1}$, then $\tau_k \circ i_k | \tilde{\mathcal{O}}_k(p)$ is the identity.

local surjectivity. The image of H_k has nonempty interior if, and only if, there exists an open set of points where the rank of H_k over the reals equals $(k+1)\mu$. This is equivalent to H_k having rank $(k+1)\mu$ over $\mathcal{K}_k$, which is equivalent to (a). Turning to almost everywhere local injectivity, (d) $\Longrightarrow$ (e) is evident. (e) $\Longleftrightarrow$ (f) is Corollary A.2 in Appendix A. It is now shown that (e) $\Longrightarrow$ (d): Applying Δ to both sides of (e) yields

$$du[1] \subset \operatorname{span}\{dx[1], dy[1], \ldots, dy[n+1]\} .$$

Hence, adding $\operatorname{span}\{dx, du[0], dy[0]\}$ to the right-hand side,

$$du[1] \subset \{dx, du[0], dy[0], dy[1], \ldots, dy[n+1]\}$$
$$\subset \operatorname{span}\{dx, dy[0], \ldots, dy[n+1]\},$$

where (e) has been used in the last step. The remainder of a proof by induction is clear. To finish up, it suffices now to show that (d) is equivalent to almost everywhere local injectivity. Without loss of generality, it can be assumed that the neighborhood $\tilde{\mathcal{O}}_k$ is such that $E_{n+k} \circ i_k : \tilde{\mathcal{O}}_k \to X \times Y^{n+k+1}$ has constant rank. Then almost everywhere local injectivity means that this rank equals $n + (k+1) \cdot m$. This is equivalent to $\operatorname{rank}_{\mathcal{K}_{n+k}} (\partial(y[0], \ldots, y[n+k])/\partial(u[0], \ldots, u[k])) = (k+1)m$, which is equivalent to (d). $\quad\square$

Remark. A quite different approach to invertibility is taken in [10]; the results of the next section show that, in the case of polynomial systems, the two approaches coincide.

5. Difference algebra and the transformal transcedence degree. The purpose of this section is to prove that the rank ρ^* defined in (2.10), when specialized to systems whose right-hand side depends polynomially on x and u (more precisely, on their components), corresponds to the transformal transcendence degree used in [10].

Consider a system

$$(5.1) \qquad \Sigma_{P,Q} \quad \begin{aligned} x[k+1] &= P(x[k], u[k]) \\ y[k] &= Q(x[k], u[k]), \end{aligned}$$

where $P : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $Q : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^\mu$ are such that each of their components is a polynomial of x and u, with coefficients in $\mathbb{R}$. This system is clearly analytic, so the analysis of §2 applies. For the purpose of clarity in presenting the results, it will be assumed that (5.1) is generically submersive. An extension to rational systems could also be undertaken.

The following definition, adapted from [10], should actually be *derived* by constructing the difference field associated to $\Sigma_{P,Q}$ and then applying the definition used in [10].

DEFINITION 5.1. The transformal transcendence degree of $\Sigma_{P,Q}$, denoted $d^0(\Sigma_{P,Q})$ equals the maximal number of components of y, say $\{y_{i_1}, \ldots, y_{i_\rho}\}$, such that for any $k \geq 0$ there does not exist any nontrivial polynomial π with coefficients in $\mathbb{R}$ such that

$$\pi\left(y_{i_1}[0], \ldots, y_{i_\rho}[0], \ldots, y_{i_1}[k], \ldots, y_{i_\rho}[k]\right) = 0.$$

In other words, for any $k \geq 0$, $y_{i_1}[0], \ldots, y_{i_\rho}[0], \ldots, y_{i_1}[k], \ldots, y_{i_\rho}[k]$ viewed as polynomials of $x, u[0], \ldots, u[k]$ are algebraically independent.

For those readers familiar with Kähler differentials, the equality $\rho^* = d^0(\Sigma_{P,Q})$ is immediate from Theorem 3.4. For the benefit of other readers, an independent, straightforward proof is given. Lemma 5.2, which follows, is well-known, though hard to find in the form presented (cf. [28]).

Let $v = (v_1, \ldots, v_r)$ be an r-tuple of indeterminants, let $I\!R[v]$ denote the ring of polynomials of $(v_1, \ldots, v_r)$ with coefficients in $I\!R$ and let $I\!R(v)$ be the corresponding field of rational functions. Define a vector space over $I\!R(v)$ by $V := \mathrm{span}\{dv_1, \ldots, dv_r\}$ and define the mapping $d : I\!R(v) \to V$ by

$$(5.2) \qquad d\left(\frac{p(v)}{q(v)}\right) := \frac{1}{(q(v))^2} \sum_{j=1}^{r} \left(\frac{\partial p(v)}{\partial v_j} q(v) - p(v) \frac{\partial q(v)}{\partial v_j}\right) dv_j$$

in the usual way; see [30, Chap. 5, §10.5] for how to define a differential calculus of rational functions without taking limits.

LEMMA 5.2. *A collection of polynomials* $\{P_1, \ldots, P_k\} \subset I\!R[v]$ *is algebraically independent if, and only if, the set* $\{dP_1, \ldots dP_k\}$ *is linearly independent in* $(V, I\!R(v))$.

Proof. Suppose that $\{P_1, \ldots, P_k\}$ is algebraically independent; then $k \leq r$. Assume first that $k = r$. $\{P_1, \ldots P_k\}$ is then a basis for $I\!R[v]$, and consequently, for each $1 \leq i \leq r$, there is a nontrivial polynomial $Q_i(\lambda_1, \ldots, \lambda_{r+1})$ such that $Q_i(v_i, P_1, \ldots, P_r) = 0$.

PROPOSITION 5.3. *For each* $1 \leq i \leq r$, *the polynomial* $\partial Q_i/\partial \lambda_1$ *is nontrivial.*

Proof. Suppose it is trivial. Then $Q_i(v_i, P_1, \ldots, P_r) = Q_i(0, P_1, \ldots, P_r) =: \overline{Q}_i(P_1, \ldots, P_r)$. Thus, $\overline{Q}_i$ must be a trivial polynomial because $\{P_1, \ldots, P_r\}$ is algebraically independent. It follows that Q_i is a trivial polynomial, which contradicts its definition. □

Continuing with the proof of Lemma 5.2, since $0 = Q_i(v_i, P_1, \ldots, P_r)$,

$$(5.3)$$

$$0 = d\left(Q_i(v_i, P_1, \ldots P_r)\right) = \frac{\partial Q_i}{\partial \lambda_1}(v_i, P_1, \ldots, P_r)dv_i + \sum_{j=1}^{r} \frac{\partial Q_i}{\partial \lambda_{j+1}}(v_i, P_1, \ldots, P_r)dP_j;$$

see [30, Chap. 5] for the chain rule. By Proposition 5.3, $\partial Q_i/\partial \lambda_1$ is nontrivial, and thus,

$$(5.4) \qquad\qquad dv_i = \sum_{j=1}^{r} k_{ij} dP_j,$$

where

$$k_{ij} := \left(\frac{\partial Q_i}{\partial \lambda_1}(v_i, P_1, \ldots, P_r)\right)^{-1} \left(\frac{\partial Q_i}{\partial \lambda_{j+1}}(v_i, P_1, \ldots, P_r)\right) \in I\!R(v).$$

Thus, $\mathrm{span}\{dv_1, \ldots, dv_r\} \subset \mathrm{span}\{dP_1, \ldots, dP_r\}$, proving the linear independence of $\{dP_1, \ldots, dP_r\}$.

If $k < r$, then there exist $P_{k+1}, \ldots, P_r$ such that $\{P_1, \ldots, P_r\}$ is a basis for $I\!R(v)$. From the above, $\{dP_1, \ldots, dP_r\}$ is linearly independent, and therefore, so must be $\{dP_1, \ldots, dP_k\}$.

To prove the other direction of the lemma, suppose that $\{P_1, \ldots, P_k\}$ is algebraically dependent. Then there exists a nontrivial polynomial $Q(\lambda_1, \ldots, \lambda_k)$ with coefficients in $I\!R$ such that $Q(P_1, \ldots, P_k) = 0$. Hence, $0 = \sum_{j=1}^{k} \partial Q/\partial \lambda_j (P_1, \ldots P_k)dP_j$,

proving that $\{dP_1, \ldots, dP_k\}$ is a linearly dependent set in $(V, \mathbb{R}(v))$. This completes the proof of Lemma 5.2.   □

The constructions and results of §2 hold clearly for polynomial systems (5.1) with the field $\mathcal{K}$ replaced[4] by $\mathbb{R}[x, u[0], \ldots, u[n]]$. This observation, combined with Lemma 5.2 and Theorem 3.4, yields the following result.

THEOREM 5.4. *For the polynomial system* (5.1), $\rho^* = d^0(\Sigma_{P,Q})$.

Remark. Lemma 5.2 can be equivalently stated as: The following two conditions are equivalent:

(a) There exists a nontrivial polynomial π such that $\pi(P_1, \ldots, P_r) = 0$

(b) The set $\{dP_1, \ldots dP_r\}$ is linearly dependent in $(V, \mathbb{R}(v))$

The implication (a) $\Longrightarrow$ (b) remains true with π, $P_1, \ldots, P_r$ replaced by analytic functions. However, the converse is then true only *locally*, and even then only on subsets where certain constant dimensional conditions are met; indeed, this is the well-known Rank Theorem.

6. Relation to generic observation fields. The chain of subspaces $\mathcal{E}_0 \subset \cdots \subset \mathcal{E}_k \subset \cdots$ is measuring how the input components are appearing in the outputs, assuming that the initial state is known. In a similar manner, one could study how the initial state components appear in the outputs, assuming that the inputs are known. This is called *observability* [15], and in the context of the formalism of this paper, could be studied via the chain $\mathcal{O}_0 \subset \cdots \subset \mathcal{O}_k \subset \cdots$, where

$$(6.1) \qquad \mathcal{O}_k := \mathrm{span}_{\mathcal{K}_k} \{du[0], \ldots, du[k], dy[0], \ldots, dy[k]\} \ .$$

Then, as in [4] and this paper, a connection could be established between the ranks of certain Jacobian matrices, the dimensions of the subspaces $\mathcal{O}_k$ and/or the transcendence degree of a certain (differential) field. This (plus a whole lot more, such as the construction of a realization theory for polynomial input-ouput maps) was done by Sontag [28] in 1979 for a very general class of discrete-time polynomial systems (cf. his generic observation fields, Q_f^K). An extension to analytic systems would follow along the lines of [31]. More recent work on the analysis of the observability of continuous-time systems by algebraic means can be found in [5] and the references therein.

Appendix A.

Proof of Lemma 2.2. For the sake of the study of invertibility in §3, it is useful to prove a little more than is required by the lemma. The following notation is used only in Appendix A; it will help to keep the formulas concise. If M is a subset of $\mathcal{E}$, then $[M]$ denotes its *span* (see [20, p. 16]). If $\{v_1, \ldots, v_s\}$ is a set of *linearly independent* elements in $\mathcal{E}$, this will be denoted by $\{v_1, \ldots, v_s\}^*$. For example, let $I_0 \subset \{1, \ldots, \mu\}$ be such that $\{dx, dy_{i_0}[0]|i_0 \in I_0\}$ is a basis for $\mathcal{E}_0$. Then, this can be succinctly stated as $[\{dx, dy_{i_0}[0]|i_0 \in I_0\}^*] = \mathcal{E}_0$.

Suppose that for some $0 \leq k$, index sets $I_0 \subset I_1 \subset \cdots \subset I_k \subset \{1, \ldots, \mu\}$ and $\{1, \ldots, m\} \supset J_0 \supset J_1 \supset \cdots \supset J_k$ have been selected so that for each $0 \leq t \leq k$,

$$(A.1) \qquad \mathcal{E}_t = [\{dx, dy_{i_r}[r]|0 \leq r \leq t \ , \ i_r \in I_r\}^*]$$

$$(A.2) \ [dx, du[0], \ldots, du[t]] = [\{dx, dy_{i_r}[r], du_{j_{t-r}}[r]|0 \leq r \leq t, i_r \in I_r, \ j_r \in J_r\}^*].$$

---

[4] So that the proof of Lemma 2.4, which used the Rank Theorem, can be carried over, one must note that (2.21) holds for (5.1) over $\mathcal{K}$ if, and only if, it holds over $\mathbb{R}[x, u[0], \ldots, u[n]]$.

and

(A.3) $$du[0] \subset \mathcal{E}_t \oplus [\{du_j[0] | j \in J_t\}].$$

That this is possible for $k = 0$ is obvious. It will first be shown that $I_{k+1} \supset I_k$ can be chosen so that (A.1) holds for $0 \le t \le k + 1$.

CLAIM A.1. *The set*

$$\left\{ dx, dy_{i_r}[r], dy_{i_k}[k+1], du_{j_k}[0], du_{j_{k-r}}[r+1] | 0 \le r \le k, i_r \in I_r, \; j_r \in J_r \right\}$$

is linearly independent.

Proof. To see how the arguments go, consider first (A.2) for $t = 0$ and apply Δ to both sides to obtain

$$[dx[1], du[1]] = [\{dx[1], dy_{i_0}[1], du_{j_0}[1] | i_0 \in I_0, \; j_0 \in J_0\}].$$

Adding $[dx, du[0]]$ to both sides yields

$$[dx, du[0], du[1]] = [\{dx, du[0], dy_{i_0}[1], du_{j_0}[1] | i_0 \in I_0, \; j_0 \in J_0\}].$$

Applying (A.2) for $t = 0$ results in

$$[dx, du[0], du[1]] = [\{dx, dy_{i_0}[0], du_{j_0}[0], dy_{i_0}[1], du_{j_0}[1] | i_0 \in I_0, \; j_0 \in J_0\}].$$

The independence of the vectors on the left-hand side implies the independence of those on the right-hand side by counting the number of elements. In particular, the vectors $\{dx, dy_{i_0}[0], dy_{i_0}[1], | i_0 \in I_0,\}$ are linearly independent, and thus, one can choose $I_1 \supset I_0$ such that (A.1) holds for $0 \le t \le 1$.

In general, consider (A.2) for $t = k$ and apply Δ to both sides to obtain

(A.4)
$$[dx[1], du[1], \ldots, du[k+1]]$$
$$= [\{dx[1], dy_{i_r}[r+1], du_{j_{k-r}}[r+1] | 0 \le r \le k, i_r \in I_r, j_r \in J_r\}].$$

Add $[dx, du[0]]$ to both sides to obtain

$$[dx, du[0], \ldots, du[k+1]]$$
$$= [\{dx, du[0], dy_{i_r}[r+1], du_{j_{k-r}}[r+1] | 0 \le r \le k, i_r \in I_r, j_r \in J_r\}^*].$$

The independence of the vectors on the left-hand side implies the independence of those on the right-hand side by counting the number of elements. Applying (A.2) first for $t = 0$, and then successively for $t = 1, \ldots, t = k$ results in

$$[dx, du[0], \ldots, du[k+1]]$$
$$= [\{dx, dy_{i_0}[0], du_{j_0}[0], dy_{i_r}[r+1], du_{j_{k-r}}[r+1] | 0 \le r \le k, i_r \in I_r, j_r \in J_r\}^*]$$
$$= [\{dx, dy_{i_0}[0], dy_{i_1}[1], du_{j_1}[0], dy_{i_s}[s+1], du_{j_{k-r}}[r+1] | 0$$
$$\le r \le k, 1 \le s \le k, i_s \in I_s, j_r \in J_r\}^*].$$
(A.5)
$$= [\{dx, dy_{i_0}[0], dy_{i_1}[1], dy_{i_2}[2], du_{j_2}[0], dy_{i_s}[s+1], du_{j_{k-r}}[r+1] | 0$$
$$\le r \le k, 2 \le s \le k, i_r \in I_r, j_r \in J_r\}^*]$$
$$= \text{for each } 0 \le t \le k$$
$$= [\{dx, dy_{i_0}[0], \ldots, dy_{i_t}[t], du_{j_t}[0], dy_{i_s}[s+1], du_{j_{k-r}}[r+1] | 0$$
$$= \qquad \le r \le k, t \le s \le k, i_r \in I_r, j_r \in J_r\}^*]. \qquad \square$$

Hence, we can choose $I_{k+1} \supset I_k$ so that (A.1) holds for each $0 \leq t \leq k+1$. It is next shown that $J_{k+1} \subset J_k$ can be chosen such that (A.2) and (A.3) hold. From (A.3) for $t = k$, since $\mathcal{E}_k \subset \mathcal{E}_{k+1}$, we deduce

(A.6)
$$du[0] \subset \mathcal{E}_{k+1} + \{du_j[0] | j \in J_k\}.$$

By the definition of I_{k+1},

(A.7) $\quad \mathcal{E}_{k+1} = \mathcal{E}_k \oplus [\{dy_{i_k}[k+1] | i_k \in I_{k+1}\}^*] \oplus [\{dy_\ell[k+1] | \ell \in I_{k+1} \backslash I_k\}^*].$

For each $\ell \in I_{k+1} \backslash I_k$, $dy_\ell[k] \in \mathcal{E}_k$, and thus,

$$
\begin{aligned}
dy_\ell[k+1] &\in [\{dx[1], dy_{i_0}[1], \dots, dy_{i_k}[k+1] | i_r \in I_r, 0 \leq r \leq k\}] \\
&\subset (\mathcal{E}_k \oplus [\{dy_{i_k}[k+1] | i_k \in I_k\}^*]) + [du[0]] \\
&\subset (\mathcal{E}_k \oplus [\{dy_{i_k}[k+1] | i_k \in I_k\}^*]) + [du[0] | j \in J_k],
\end{aligned}
$$

(A.8)

where the last inclusion is from (A.3) for $t = k$. Thus, $(|I_{k+1}| - |I_k|)$ elements of $\{du_j[0] | j \in J_k\}$ are not independent of $\mathcal{E}_{k+1}$. One can therefore choose $J_{k+1} \subset J_k$ such that

(A.9)
$$du[0] \subset \mathcal{E}_{k+1} \oplus [\{du_j[0] | j \in J_{k+1}\}^*]$$

and

(A.10)
$$|J_{k+1}| = |J_k| - (|I_{k+1}| - |I_k|).$$

To finish up, from (A.5) and (A.9) it follows that

$$[dx, du[0], \dots du[k+1]] \subset [\{dx, dy_{i_0}[0], \dots, dy_{i_{k+1}}[k+1], du_{j_{k+1}}[0], \dots,$$

(A.11)
$$du_{j_0}[k+1] | i_t \in I_t, j_{k+1-t} \in J_t, 0 \leq t \leq k+1\}].$$

Since the reverse inclusion is obviously true, one has equality. By counting the number of vectors on the right-hand side of (A.11), we obtain (A.2) for $t = k+1$. $\quad \Box$

From (A.1) and (A.2), respectively, it follows that

(A.12)
$$\rho_k = n + \sum_{i=1}^{k} |I_i|$$

and

(A.13)
$$n + (k+1)m = n + \sum_{i=1}^{k} |I_i| + \sum_{i=1}^{k} |J_i|,$$

yielding

(A.14)
$$\rho_k - \rho_{k-1} = m - |J_k|.$$

This combined with (A.3) proves the following result.

COROLLARY A.2. *Suppose that (2.1) is generically submersive. Then*

$$\dim \mathcal{E}_k - \dim \mathcal{E}_{k-1} = m \ \textit{if and only if} \ \{du_1[0], \dots, du_m[0]\} \subset \mathcal{E}_k \ .$$

Proof of Lemma 2.4. We can view $y[k]$, $0 \le k \le N$ as being an analytic function on $X \times U^{N+1}$. Let $\mathcal{O}$ be an open subset of $X \times U^{N+1}$.

By the definition of $I_0, \dots, I_n$, for any $0 \le N \le n$,

$$(A.15) \qquad dy_j[N] \in \mathrm{span}_{\mathcal{K}_N} \{dx, dy_{i_0}[0], \dots, dy_{i_N}[N] | i_k \in I_k, \ 0 \le k \le N\} \ .$$

Due to analyticity, (A.15) is equivalent to

$$(A.16) \ \ dy_j[N]|_{\mathcal{O}} \in \mathrm{span}_{\mathbb{R}} \{dx, dy_{i_0}[0], \dots, dy_{i_N}[N] | i_k \in I_k, \ 0 \le k \le N\}|_{\mathcal{O}} \ ,$$

where the left-hand side is viewed as a one-form on $\mathcal{O}$, the right-hand side is viewed as an analytic codistribution on $\mathcal{O}$, and the span is taken pointwise; without loss of generality, it can be assumed that the codistribution has constant dimension. After possibly shrinking $\mathcal{O}$, the Rank theorem implies that on $\mathcal{O}$, $y_j[N]$ can be expressed as an analytic funtion of $(x, y_{i_0}[0], \dots, y_{i_N}[N] | i_k \in I_k, \ 0 \le k \le N)$. Repeating the above reasoning, (2.21) is equivalent to

$$(A.17) \qquad dy_j[N]|_{\mathcal{O}} \in \mathrm{span}_{\mathcal{K}_N} \{dy_j[0], \dots, dy_j[N-1], dy_{i_0}[0],$$
$$\dots, dy_{i_N}[N] | i_k \in I_k, \ 0 \le k \le N\}|_{\mathcal{O}}.$$

Hence, from (A.16), (2.21) holds if

$$(A.18) \qquad \frac{\partial y_j[N]}{\partial x} dx \in \ \mathrm{span}_{\mathcal{K}_N} \left\{ \frac{\partial y_j[0]}{\partial x} dx, \dots, \frac{\partial y_j[N-1]}{\partial x} dx \right\} \ .$$

Because the right-hand side of (A.18) can be at most n-dimensional, there must exist an N, $1 \le N \le n$, such that this is the case. $\square$

Appendix B.

Proofs of Lemmas 2.6 and 2.7. The inclusion $M_{k+1} \subset M_k$, implies $d_{k+1} \le d_k \ \forall k \ge 0$. Since $x[k] : \mathbb{R}^n \times (\mathbb{R}^m)^k \to \mathbb{R}^n$ is an analytic function, there exists an open and dense subset $V_k \subset \mathbb{R}^n \times (\mathbb{R}^m)^k$ on which $x[k]$ has constant $\mathbb{R}$-rank equal to d_k; that is, for each point $p \in V_k$,

$$(B.1) \qquad \mathrm{rank}_{\mathbb{R}} \left[\frac{\partial x[k]}{\partial (x, u[0], \dots, u[k-1])} \right] (p) = d_k \ .$$

Let $N_k := x[k](V_k)$. Then, N_k is an immersed submanifold of $\mathbb{R}^n$, but since the resulting topology may be different than the subset topology of $\mathbb{R}^n$, this property is of little interest. More importantly, the implicit function theorem implies that, for each point $q \in N_k$, there exists an open subset $\mathcal{O}_k \subset V_k$ such that $x[k](\mathcal{O}_k)$ is a d_k-dimensional, embedded, analytic submanifold of $\mathbb{R}^n$ and q is in the interior of $x[k](\mathcal{O}_k)$. Since V_k is dense in $\mathbb{R}^n \times (\mathbb{R}^m)^k$, N_k is dense in M_k in the subset topology.

Let $\overline{f}_k : x[k](\mathcal{O}_k) \times \mathbb{R}^m \to \mathbb{R}^n$ denote f restricted to $x[k](\mathcal{O}_k) \times \mathbb{R}^m$ (the subscript k is to note that $\overline{f}_k$ depends on $\mathcal{O}_k$). Whenever $\mathcal{O}_k \subset V_k$ is such that $x[k](\mathcal{O}_k)$ is a d_k-dimensional embedded submanifold of $\mathbb{R}^n$, then $\overline{f}_k$ is an analytic function. Consequently, it will have constant $\mathbb{R}$-rank on an open and dense subset of its domain

of definition, which is called its *generic rank* and is denoted as gen rank $\overline{f}_k$. From (2.23) and (2.24) it follows that

(B.2) $$d_{k+1} = \text{gen rank} \overline{f}_k : x[k](\mathcal{O}_k) \times I\!\!R^m \to I\!\!R^n$$

since, if $p \in \mathcal{O}_k$ and $q := x[k](p)$, then $T_q x[k](\mathcal{O}_k)$, the tangent space of $x[k](\mathcal{O}_k)$ at the point q, satisfies

(B.3) $$T_q x[k](\mathcal{O}_k) = \text{Image} \left[\frac{\partial x[k]}{\partial(x, u[0], \dots, u[k-1])} \right](p).$$

With the above preliminaries completed, the proof of Lemma 2.6 can be given. If $d_n = 0$, the result is obvious; suppose, therefore, that $d_n > 0$. Then, there exists $0 \le k \le n - 1$ such that $d_k = d_{k-1}$. As before, choose $\mathcal{O}_k$ to be an open subset of V_k such that $x[k](\mathcal{O}_k)$ is an embedded d_k-dimensional submanifold of $I\!\!R^n$. Since $M_k \subset M_{k-1}$, and N_k and N_{k-1} are dense in M_k and M_{k-1}, respectively, the condition $d_k = d_{k-1}$ implies that $x[k](\mathcal{O}_k)$ is an embedded $(d_{k-1} = d_k)$-dimensional submanifold of $I\!\!R^n$ and $x[k](\mathcal{O}_k) \cap x[k-1](\mathcal{O}_{k-1})$ has nonempty interior. Therefore,

(B.4) $$\begin{aligned} d_{k+1} &= \text{gen rank} \overline{f}_k : x[k](\mathcal{O}_k) \times I\!\!R^m \to I\!\!R^n \\ &= \text{gen rank} \overline{f}_{k-1} : x[k-1](\mathcal{O}_{k-1}) \times I\!\!R^m \to I\!\!R^n \\ &= d_k \, , \end{aligned}$$

where the fact that $\overline{f}_k$ and $\overline{f}_{k-1}$ are the restrictions of a common map f and have a common nonempty open set in their domain of definition entails the second equality. This completes the proof of Lemma 2.6.

Turning to Lemma 2.7, let $d := d_n$, which is then equal to d_k for all $k > n$ by Lemma 2.6. If $d = 0$, then Lemma 2.7 is immediate, so in the following it is supposed that $d > 0$. Let $\alpha = \alpha(x, u[0], \dots, u[k])$ be an element of $\mathcal{R}_k$. For any $r > 0$, $\delta^r(\alpha) = \alpha(x[r], u[r], \dots, u[k+r])$. Let $\tilde{\mathcal{O}}_r \subset V_r$ be an open set such that $x[r](\tilde{\mathcal{O}}_r)$ is a d_r-dimensional submanifold of $I\!\!R^n$. Then $\delta^r(\alpha) \not\equiv 0$ if, and only if, α restricted to $(x[r](\tilde{\mathcal{O}}_r) \times (I\!\!R^m)^{k+1}) \not\equiv 0$. When $r > n$, there exists $\tilde{\mathcal{O}}_n \subset V_n$ such that $x[n](\tilde{\mathcal{O}}_n) \times (I\!\!R^m)^{k+1} \cap x[r](\tilde{\mathcal{O}}_r) \times (I\!\!R^m)^{k+1}$ has nonempty interior. Thus, α restricted to $(x[r](\tilde{\mathcal{O}}_r) \times (I\!\!R^m)^{k+1}) \not\equiv 0$ if, and only if, α restricted to $(x[n](\tilde{\mathcal{O}}_n) \times (I\!\!R^m)^{k+1}) \not\equiv 0$. ∎

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