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FEEDBACK LINEARIZATION OF DISCRETE-TIME SYSTEMS[†]

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Abstract

Necessary and sufficient conditions are given for a nonlinear discrete-time system to be feedback equivalent to a controllable linear system. Some preliminary work on the effects of sampling on feedback linearizability is reported.

I. Introduction

The problem of determining when a nonlinear continuous-time system can be transformed into a linear system by changes of state coordinates and (nonsingular) feedback has been extensively studied (local results, Hunt and Su 1981, Jakubczyk and Respondek 1980, Krener 1973; global results, Cheng et al. 1985, Dayawnsa et al. 1985, Respondek 1985). This paper will address the corresponding local problem for nonlinear discrete-time systems of the form:

$$\Sigma : x_{k+1} = f\,(x_k, u_k)\,; \tag{1.1}$$

here $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and f is assumed to be an analytic function of its arguments. Necessary and sufficient conditions will be given for the local existence of a new set of coordinates $\tilde{x} = \phi(x)$ and a nonsingular feedback $u = \gamma(\tilde{x}, \tilde{u})$, $\det(\frac{\partial \gamma}{\partial \tilde{u}}) \neq 0$, which transform (1.1) into a controllable linear system

$$\tilde{x}_{k+1} = A\tilde{x}_k + B\tilde{u}_k\ . \tag{1.2}$$

The result will closely parallel that of van der Schaft 1985 for (general) nonlinear continuous-time systems, which is in turn based on Jakubczyk and Respondek 1980.

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In practice, many (if not most) discrete-time systems arise by sampling continuous-time systems. It is, therefore, of interest to investigate how certain properties of continuous-time systems are affected by sampling. Considerable work has been done in this area by Sontag who has studied in detail the preservation of controllability (Sontag 1985) and observability (Sontag 1984a) under sampling. Monaco and Normand-Cyrot 1985 have investigated how invariant distributions behave under approximate sampling procedures. They have also given an explicit power series (in the input variables) representation of a sampled system in terms of the defining vector fields of the original continuous-time control system (Monaco and Normand-Cyrot 1985b,c). However, even in the case of linear systems, the author is unaware of any results on how the solvability of the various so-called synthesis problems is affected by sampling. Some results in this direction for linear systems will appear in Shor 1986. Here, some preliminary work on the effects of sampling on feedback linearizability will be reported.

In work related to the feedback linearization problem, those nonlinear discrete-time systems which can be transformed into controllable linear systems, *plus* terms of order higher than some specified integer, have been characterized by Lee and Marcus 1985. They also give a result on the (exact) feedback linearization problem, but it is uncheckable.

A less restrictive notion than feedback linearization involves immersing a nonlinear system into a linear system (see Claude et al. 1983). This has been studied by Monaco and Normand-Cyrot 1983 for those discrete-time systems which are linear in the control:

$$x_{k+1} = f(x_k) + \sum_{i=1}^{m} u_k^i g^i(x_k). \tag{1.3}$$

A similar result, using much different techniques, is given by Sontag 1984b.

Finally, Jackubczyk has informed the author that he has obtained necessary and sufficient conditions for local feedback linearization using techniques introduced in Jakubczyk and Normand-Cyrot 1984.

II. Preliminaries and Definitions

In the following

$$\Sigma : x_{k+1} = f(x_k, u_k)$$

will always denote the discrete-time system given in (1.1). Feedback will be given by an analytic function $\gamma : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^m$ such that for each $x \in \mathbf{R}^n$, $\gamma(x,\cdot) : \mathbf{R}^m \to \mathbf{R}^m$ is a diffeomorphism. As a point of notion, *subscripts* will be used to denote the value of a vector at a particular time instant and *supersripts* will be used to denote the components of a vector or of a vector-valued function.

Definition 2.1

System (1.1) is (internally) *feedback linearizable* if there exist a feedback γ and global coordinates $\tilde{x}$ for $\mathbf{R}^n$,$\tilde{u}$ for $\mathbf{R}^m$ in which

$$f\,(x\,,\gamma(\tilde{x}\,,\tilde{u}\,)) = A\tilde{x} + B\tilde{u} \tag{2.1}$$

for some constant matrices A and B.

Note that if a system is feedback linearizable, there necessarily exists a pair of points $(x_0,u_0) \in \mathbf{R}^n \times \mathbf{R}^m$ such that $f\,(x_0,u_0) = x_0 : (x_0,u_0)$ will be called an *operating point*. It is henceforth assumed that (1.1) possesses an operating point.

Making precise a local version of the above definition results in the following.

Definition 2.2

System (1.1) is (internally) *locally feedback linearizable* about an operating point (x_0,u_0) if:

a) There exist open sets $\mathbf{O}_1 \subset \mathbf{O}_2$ about x_0 and open sets $U_1 \subset U_2$ about u_0 such that $f\,:\mathbf{O}_1 \times U_2 \to \mathbf{O}_2$.

b) There exist a *nonsingular feedback* $\gamma : \mathbf{O}_1 \times U_1 \to U_2$ and local coordinates $\tilde{x}$ and $\tilde{u}$ defined on $\mathbf{O}_2$ and U_2 in which

$$f\,(x\,,\gamma(\tilde{x}\,,\tilde{u}\,)) = A\tilde{x} + B\tilde{u}$$

for all $\tilde{x} \in \mathbf{O}_1$ and $\tilde{u} \in U_1$. Here, γ is said to be nonsingular if for each $\tilde{x} \in \mathbf{O}_1$. $\gamma(\tilde{x}\,,\cdot) : U_1 \to U_2$ is one-to-one.

It is this latter property which will be characterized. In the proof of the main result, it will be convenient to view $\pi : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ as a fiber bundle, where π is the canonical projection. In this context, feedback is simply a state-dependent change of the input coordinates and Definition 2.2 can be restated as follows:

Definition 2.3

The system (1.1) is locally feedback linearizable about an operating point (x_0,u_0) if there exist fiber respecting coordinates $(\tilde{x}\,,\tilde{u}\,)$ about (x_0,u_0) in which

$$f\,(\tilde{x}\,,\tilde{u}\,) = A\tilde{x} + B\tilde{u}\ .$$

The following technical result, proved in van der Schaft 1982, will be needed in the next section.

Lemma 2.1

Let $h : M \to N$ be a smooth function, from the smooth manifold M to the smooth manifold N, such that $h_* : TM \to TN$ is surjective. Let $K := \ker h_*$, and let D be an involutive distribution on M such that $D \cap K$ has constant dimension. Then $h_* D$ is a well-defined involutive distribution on N if and only if D+K is involutive.

III. Local Linearizability

This section states and proves necessary and sufficient conditions for local linearizability. When a system is linear, its attainable set up to time k, $A_k(x) :=$ the set of points in $\mathbb{R}^n$ which can be attained in k steps of fewer starting from an initial state x, has a very special structure. Namely, for each $k \geq 1$, $A_{k-1}(x)$ is imbedded nicely in $A_k(x)$. In particular, for all $k \geq 0$, $x \in \mathbb{R}^n$, the tangent space of $A_k(x)$ at x exists and gives rise to a well-defined *regular* (i.e., involutive and constant dimensional) distribution $\Delta_k(x) := T_x A_k(x)$. For each k, $\Delta_{k+1} \supset \Delta_k$, and if the system is controllable, Δ_n has dimension n since $A_n(x) = \mathbb{R}^n$. The basic idea of Theorem 3.1 is to use the above geometric structure in order to obtain a characterization of local feedback linearizability.

Theorem 3.1

Let Σ be a discrete-time nonlinear control system of the form (1.1) with operating point (x_0, u_0). Then the following are equivalent.

(a) Σ is locally linearizable about (x_0, u_0) to a controllable system.

(b) $f_*(x_0, u_0)$ has full rank and there exists an open set $\mathbf{O}$ about (x_0, u_0) such that $D_n = n + m$ where

$$D_0 = \pi_*^{-1}(0) | \mathbf{O}$$

$$D_{i+1} = \begin{cases} \pi_*^{-1} \tilde{f}_*(D_i) | \mathbf{O} & \text{if } D_i + K \textit{ is involutive} \text{ and } D_i \cap K \textit{ has constant dimension} \\ D_i & \textit{otherwise} \end{cases}$$

and $\tilde{f} := f | \mathbf{O}$, $K := \ker f_*$.

(c) There exists an open set $\mathbf{O}$ about (x_0, u_0) and regular distributions $D_0, D_1, \cdots, D_n$ defined about (x_0, u_0) such that $D_0 = \pi_*^{-1}(0) | \mathbf{O}$, $D_{i+1} = \pi_*^{-1} \tilde{f}_*(D_i) | \mathbf{O}$, and $\dim D_n = n + m$, where $\tilde{f} = f | \mathbf{O}$.

Remark 3.1

The importance of (b) is that it identifies the local obstructions to linearizability for example, noninvolutivity of $D_i + \ker f_*$ for some i, whereas (c) gives a geometric characterization for local linearizability, namely the existence of a sequence of nested regular distributions satisfying a certain recurrence relation.

Proof

$(a) \Rightarrow (b)$: By (a), there exist fiber respecting coordinates $(\tilde{x},\tilde{u})$ about (x_0,u_0) in which Σ has the form (2.1). In these coordinates, $f_*(x_0,u_0)=[A \mid B]$ which must have full rank for (2.1) to be controllable. One easily calculates that $D_0 = span\{\frac{\partial}{\partial \tilde{u}}\}$, $D_i = \mathbf{B} + \cdots + A^{i-1}\mathbf{B}x\ span\{\frac{\partial}{\partial \tilde{u}}\}$ where $\mathbf{B} = ImB$. Hence, the indicated involutivity and constant dimension conditions hold. Finally, $dim\ D_n = n+m$ follows from the controllability of (2.1).

$(b) \Rightarrow (c)$: Lemma 2.1 implies that each D_i is a well-defined and regular distribution. Moreover, the hypothesis that $dim\ D_{n+m} = n$ implies that each D_i satisfies $D_{i+1} = \pi_*^{-1}\tilde{f}_*(D_i) \mid \mathbf{O}$.

$(c) \Rightarrow (a)$: Since $f(x_0,u_0) = x_0$, given any fiber respecting coordinate chart (V,ϕ) about (x_0,u_0), there always exists an open set $\mathbf{O} \subset V$ such that $f(\mathbf{O}) \subset \pi(V)$. Thus one may work in a single coordinate chart. Define $\Delta_i = \pi_* D_i$, $\mu_i = dim\ \Delta_i$, and set $p_1 = \mu_1$, $p_i = \mu_i - \mu_{i-1}$ for i=1,2,...,n. Since $D_0 \subset ker\ \pi_* \subset D_i$ for each i, Lemma 1 implies that each D_i is a regular distribution on **M**. By induction, one can show that $\Delta_i \subset \Delta_{i+1}$. From the fact that $dim\ D_n = n+m$, one obtains the existence of a first integer $N \leqslant n$ such that $dim\ \Delta_N = n$. Therefore, $\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_N$ is a nested sequence of distribution which can be simultaneously integrated (Respondek and Jakubczyk 1980), to yield a coordinate system $(x^1, \cdots, x^N)$ about x_0, each x^i being a vector of dimension p_i, such that

$$\Delta_i = span \left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^i} \right\} \quad i = 1,\ldots,N.$$

Let (x,u) be fiber respecting coordinates, with $x = (x^1, \ldots, x^N)$, and write $f = (f^1, \ldots, f^N)$ in the obvious way.

Lemma 3.1

A. The component functions f^j do not depend upon the variables $x^i, \ldots, x^{j-2}$ for j=3,4,...,N, nor does f^j depend upon u for $j \geqslant 2$.

B. Rank $\frac{\partial f^j}{\partial x^{j-1}} = p_j$ for $j \geqslant 2$, and rank $\frac{\partial f^1}{\partial u} = p_1$.

C. $p_{j-1} \geqslant p_j$ for j=2,...,N.

The proof of the lemma will be delayed until the proof of the theorem is complete. As in Jakubczyk and Respondek 1980 and in van der Schaft 1984, the rest of the proof consists of successively modifying the coordinates $(x^1, \ldots, x^N)$ in such a way that A and B of Lemma 3.1 continue to hold, and that the component functions f^i become "projections."

The first step is to introduce the new coordinates $y_{N-1} = (f^N, \tilde{x}^{N-1})$, $y^j = x^j$ if $j \neq N-1$, where $\tilde{x}^{N-1}$ are $(p_{N-1} - p_N)$ components of x^{N-1} chosen in such a way that $rank\ \frac{\partial y^{N-1}}{\partial x^{N-1}} = p_{N-1}$; this is possible by B. Relabeling $(y^1, \ldots, y^N)$ once again as $(x^1, \ldots, x^N)$, f can be written as $(f^1, \ldots, f^{N-2}, \bar{x}^{N-1})$, where $\bar{x}^{N-1}$ denotes the first p_N coordinates of x^{N-1}; in other words, in these coordinates, f^N is the projection onto the first p_N of the x^{N-1} coordinate components. Using A, it is clear that A a$(x^1, \ldots, x^N)$ coordinates.

In the second step, one introduces the new coordinates $y^{N-2} = (f^{N-2}, \tilde{x}^{N-2})$, $y^j = x^j$ for $j \neq N-2$, and obtains $f = (f^1, \ldots, f^{N-2}, \bar{x}^{N-2}, \bar{x}^{N-1})$ (the bar notation continues to denote the first p_i coordinates of x^{i-1}). Continuing by induction, after N-1 steps, one has that $f(x,u) = (f^1(x,u), \bar{x}^1, \ldots, \bar{x}^{N-1})$. One now performs a state-dependent change of the inputs. Define new input coordinates by $v = (f^1(x,u), \tilde{u})$, where $\tilde{u}$ denotes $(m - p_1)$ components of u chosen such that $rank\ \frac{\partial v}{\partial u} = m$; this is posible by B. Of course, $v = \gamma(x,u) := (f^1(x,u), \tilde{u})$ can be interpreted as state-variable feedback. In the (x,v) coordinates, one finally obtains that

$$f(x,v) = (\bar{v}, \bar{x}^1, \ldots, \bar{x}^{N-1}), \tag{3.1}$$

which is clearly a controllable linear system.

□

Remark 3.2 : A simple permutation of the coordinates $(x^1, \ldots, x^N)$ can be done to put (3.1) into Brunovsky canonical form.

It remains to prove Lemma 3.1. This is done next.

Proof of Lemma 3.1

Everything will follow from the relation

$$\Delta_i = f_* D_{i-1} \mid \pi(\mathbf{O}). \tag{3.3}$$

Let (x,u) be the coordinates constructed immediately preceding the statement of the lemma. Then $f_* D_0 = \Delta_1$ in a neighborhood of x_0 implies that for $\hat{x}, \hat{u}$ near x_0, u_0,

$$span\left\{\sum_{j=1}^{N} \frac{\partial f^j}{\partial u_i}(\hat{x},\hat{u}) \frac{\partial}{\partial x^j} ; i = 1, \ldots, m\right\} = span\left\{\frac{\partial}{\partial x^1}\right\}.$$

This yields $\frac{\partial f^j}{\partial u^i} = 0$ for j=2,...,N and i=1,...,m, and also shows that $rank\ \frac{\partial f^1}{\partial u} = p_1$. Consider now $f_* D_1 = \Delta_2$ in a neighborhood of x_0. This gives

$$span\left\{\sum_{j=1}^{N} \frac{\partial f^j(\hat{x},\hat{u})}{\partial x^1} \frac{\partial}{\partial x^j},\ \frac{\partial f^1(\hat{x},\hat{u})}{\partial u} \frac{\partial}{\partial x^1}\right\} = span\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right\}.$$

which yields $\frac{\partial f^j}{\partial x^1} = 0$ for j=3,...,N and $rank\ \frac{\partial f^2}{\partial x^1} = p_2$. This last fact shows that $p_2 \leq p_1$ as p_1

equals the dimension of x^1.

The rest of the proof proceeds in a similar manner and is left to the reader.

□

IV. Linearizability and Sampling

With present technology, controllers are often implemented digitally even when their design is based upon a continuous-time (i.e., analog) system model. Such implementation necessitates making certain approximations and motivates working directly with a discrete-time (sampled-data) model of the system (Monaco and Normand-Cyrot 1985c). The question therefore arises, if a continuous-time system is feedback linearizable, are its sampled-data versions necessarily feedback linearizable for most sampling intervals T? In other words, if feedback linearization is a viable design method for a given continuous-time system, will it also be applicable to the sampled system? Note that one is only insisting upon achieving linearity from sample instant to sample instant and nothing is claimed about the interim periods.

It will be shown, via examle, that, in general, feedback linearizability is not preserved under sampling. Moreover, the example suggests the following. Given any locally linearizable continuous-time system, consider its orbit under the *feedback group* (i.e., the group of coordinate transformations and invertible state-variable feedbacks). Then the orbit contains an infinite number of elements for which linearizability will be preserved under sampling and an infinite number of elements for which it will not be preserved. Of course, the basic difficulty here is that sampling and feedback do not commute.

The example is as follows:

$$\begin{aligned} \dot{x}^1 &= x^2 \\ \dot{x}^2 &= u\,(1 + (x^2)^2) \end{aligned} \tag{4.1}$$

$x^1, x^2, u \in \mathbb{R}$. This system was chosen because its closed-form solution is easily obtained; unfortunately, it does have a finite escape time for constant u, but the reader should be able to convince himself that this is *not* the obstruction to linearizability when the system is sampled.

For a sampling interval $T > 0$ and u constant, one calculates the sampled-data system to be

$$\begin{aligned} x_{k+1}^1 &= x_k^1 + \int_0^T \tan(u\,\tau + \arctan(x_k^2))d\,\tau \\ x_{k+1}^2 &= \tan(uT + \arctan(x_k^2)) \end{aligned} \tag{4.2}$$

where $x_k := x(kT)$ (this is exact discretization). Letting $F_T(x_k, u_k)$ denote the right-hand side of (4.2), one calculates further that

$$ker\ F_{T_*} = span\{\int_0^T (\frac{\tau}{T} - 1)[1 + \tan^2(u\tau + arctan(x^2))] \frac{1}{1+(x^2)^2} d\tau \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} - \frac{1}{T}\frac{1}{1+(x^2)^2}\frac{\partial}{\partial u}\}. \tag{4.3}$$

A final calculation shows that for all $T > 0$, $ker F_{T_*} + span\{\frac{\partial}{\partial u}\}$ is not involutive, and hence (4.2) is not feedback linearizable for *any* sampling interval $T > 0$.

Since exact discretization cannot be achieved in practice, the next question is whether any approximate discretization schemes preserve linearizability? This subject will not be addressed in any measure of completeness here. It is simply remarked that, it one lets $F_T(x,u)$ denote the sampled-data system (4.2), and if one obtains approximating systems by performing a Taylor expansion in the sampling intervl T, then in the coordinates used above, neglecting second and higher order terms in T results in a system which is linearizable, whereas neglecting only third and higher order terms results in a system which is not linearizable. The first cited approximation corresponds, by the way, to an Euler integration scheme applied to the differential equation (4.1). Obviously, much work remains to be done here, and it is important to note that Taylor expansions are coordinate dependent.

V. Conclusions and Comments

Necessary and sufficient conditions for local feedback linearization of a nonlinear discrete-time system were given. In an attempt to understand some of the trade-offs involved in modeling a continuous-time system with a discrete-time model, the question of whether feedback linearizability was preserved under sampling (i.e., exact discretization) was raised. This was answered in the negative via an example which also showed that many higher-order approximate discretization schemes would probably result in systems which were also not feedback linearizable. In conclusion, the following conjecture is made.

Conjecture

Let $\Sigma : \dot{x} = f(x,u)$ be a single-input analytic control system on $\mathbb{R}^n$ such that f(0,0) = 0, and let $x_{k+1} = F_T(x_k, u_k)$ be its sampled version for sampling interval T. Then $F_T(\cdot,\cdot)$ is locally feedback linearizable for an open set of sampling times if and only if Σ is state-equivalent to a controllable linear system (i.e., Σ can be linearized using only state transformations).

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