Formal Embedding of the Spring Loaded Inverted Pendulum in an Asymmetric Hopper

Ioannis Poulakakis, and J. W. Grizzle, *Fellow, IEEE*

Abstract—**The control of running is discussed in terms of a model called the Asymmetric Spring Loaded Inverted Pendulum (ASLIP), shown in Fig. 1. The ASLIP consists of a Spring Loaded Inverted Pendulum (SLIP) with the addition of pitch dynamics, and can be used to study the sagittal plane motion of bipedal running. A hybrid controller for the ASLIP is developed that acts on two levels. In the first level, continuous in-stride control is used to stabilize the torso at a desired posture, and to create an invariant surface on which the stance dynamics of the closed-loop system is diffeomorphic to the center of mass dynamics of a SLIP. In the second level, event-based control is employed to stabilize the closed-loop hybrid system along a periodic orbit of the SLIP dynamics. These results provide a systematic framework for designing control laws with provable stability properties which take advantage of existing SLIP controllers that are known to induce elegant running motions in legged models.**

I. Introduction

Most of the hopping and running robots introduced over the past twenty years have employed controllers that are variations of Raibert's original controller, [12]. These controllers regulate forward speed by positioning the legs during the flight phase at a proper touchdown angle, while, during the stance phase, hip torque and leg thrust are employed to regulate hopping height and body attitude.

The combined difficulties of hybrid dynamics and underactuation inherent in legged systems stymied the direct application of nonlinear controller synthesis tools, such as those in [10], to running robots and led many researchers to believe that the problem did not fit well within the framework of modern nonlinear control theory. Despite this widespread belief, results in [8], [17], and [5], have demonstrated the utility of nonlinear control theory in inducing provably asymptotically stable dynamic walking and running motions in bipedal robots. In particular, it has been shown that planar walking and running gaits can be "embedded" in the dynamics of a closed-loop system by defining a set of holonomic output functions with the control objective being to drive these outputs to zero [8], [17]. In essence, this method asymptotically restricts the dynamics of the closed-loop hybrid model to a lower-dimensional attractive and invariant subset of the state space. The stable periodic solutions of the dynamics restricted on this attractive and invariant subset, called the Hybrid Zero Dynamics (HZD), encode the desired task (walking or running).

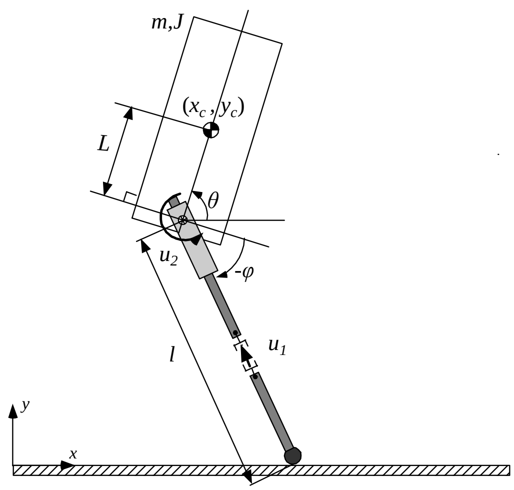

Figure 1. The Asymmetric Spring Loaded Inverted Pendulum (ASLIP).

The general idea of task encoding through the enforcement of a lower-dimensional target dynamics, rather than through the prescription of a set of reference trajectories, has been employed successfully in the control of dynamically dexterous machines, including juggling, brachiating and running robots, by Koditschek and his collaborators [3], [11] and [13]. In this work however, the restriction dynamics is specified *a priori*, through the selection of a dynamical system that is believed to capture the salient characteristics of the task, and is not the outcome of the control design procedure as in [17]. Task encoding through imposing pre-specified target dynamics leaves one with the question of selecting a suitable candidate dynamical system for the targeted behavior, and turned attention of the robotics community into models that have been inspired by biomechanics.

Faced with the complexity of studying running in animals, biologists proposed the Spring Loaded Inverted Pendulum as a template, [6]. Notwithstanding its apparent simplicity, the SLIP has been useful in (qualitatively) explaining various aspects of running in animals [6], and in designing empirical controllers for robots [12]. These findings have prompted a deeper study of the SLIP, with the aim of understanding whether the SLIP is merely one way of describing the kinematics of the observed periodic orbits, or whether it

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I. Poulakakis and J. W. Grizzle are with the Control Systems Laboratory, Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48105, U.S.A. (phone: +1-734-936-3875; fax: +1-734-763-8041; e-mail: poulakas@ eecs.umich.edu).

represents a dynamic model of the observed running behavior of animals, and thus would be an interesting target dynamics for legged robots [6], [14]. These research efforts produced a large variety of controllers for the SLIP, see [14] and references therein, and more recently [2], [7], [15], [16]. These controllers exhibit very appealing properties such as large domains of attraction and minimal control effort.

Up to this point, however, much of this research has been concentrated on the SLIP itself. The formal connection between the SLIP and more elaborate models that enjoy a more faithful correspondence to a typical robot's structure and morphology has not been fully investigated. It still remains unclear how stability conclusions obtained in the context of the SLIP can predict the behavior of more complete models; only preliminary results in this direction are available [13]. Furthermore, as was shown in [4], controllers specifically derived for the SLIP will have to be modified in order to be successful in inducing stable running in more complete models that include pitch dynamics or energy losses. These observations set the stage of this research, which aims at establishing a more formal connection between the SLIP as a control target for running and more complete plant models of legged robots. Hence, rather than analyzing the much studied SLIP (see [2], [7], [14], [15], [16] for example), we turn our attention to its implications in the control of running.

In this paper, a framework is proposed that provides a systematic procedure for designing feedback controllers with provable properties that are suitable for inducing running motions in an asymmetric hopping model. This framework combines established nonlinear control synthesis tools, such as the HZD originally proposed in [17], with controllers obtained in the context of the SLIP e.g. [2], [12], [14], [16]. Hence, given *any* controller that results in an exponentially stable periodic solution of the SLIP, the method developed here shows how to "embed" the SLIP orbit in a more complete model that includes nontrivial torso pitch dynamics. It is emphasized that the practical consequences of these results lie in the fact that they allow the direct use of controllers obtained for the SLIP in a more complete model. This model is called the *Asymmetric Spring Loaded Inverted Pendulum (ASLIP)*, and can be used to study the sagittal plane running of bipedal robots. Despite its importance, to the best of the authors' knowledge, no formal studies of the ASLIP exist. Proposing and formally analyzing control laws for the stabilization of the ASLIP that take advantage of SLIP controllers constitutes the goal of this paper.

II. The Asymmetric Spring Loaded Inverted Pendulum

A schematic for the Asymmetric Spring Loaded Inverted Pendulum (ASLIP) is presented in Fig. 1. The hip joint does not coincide with the center of mass (COM) of the torso, which is modeled as a rigid body with mass m and moment of inertia J about the COM. The leg is assumed to be massless. The ASLIP is controlled by two inputs: a force u_1 acting along the leg, and a torque u_2 applied at the hip. In what follows, the subscripts "f" and "s" denote "flight" and "stance," respectively.

A. Flight Phase Dynamics

The flight phase dynamics corresponds to a point mass undergoing ballistic motion in a gravitational field together with a double integrator governing the pitch motion. The configuration space Q_f of the flight phase is a simply-connected open subset of $\mathbb{R}^2 \times S^1$ corresponding to physically reasonable configurations of the ASLIP, and it can be parameterized by the Cartesian coordinates of the COM together with the pitch angle, i.e. $q_\mathrm{f} = (x_\mathrm{c}, y_\mathrm{c}, \theta)' \in Q_\mathrm{f}$. The flight phase dynamics of the ASLIP evolves in $TQ_\mathrm{f} = \{x_\mathrm{f} = (q_\mathrm{f}', \dot{q}_\mathrm{f}')' \mid q_\mathrm{f} \in Q_\mathrm{f}, \dot{q}_\mathrm{f} \in \mathbb{R}^3\}$, and can be easily written in state-space form

$$\dot{x}_\mathrm{f} = f_\mathrm{f}(x_\mathrm{f}). \tag{1}$$

The flight phase terminates when the vertical distance of the toe from the ground becomes zero. To realize this condition, the flight phase state vector is augmented with $a_\mathrm{f} = (l, \varphi)' \in A_\mathrm{f} \subseteq \mathbb{R} \times S^1$, where l and φ are the leg length and angle, respectively, and $\dot{a}_\mathrm{f} = 0$. This is a consequence of the assumption of a massless leg; during flight, the leg obtains the desired length and orientation instantaneously. The threshold function $H_{\mathrm{f}\to\mathrm{s}} : TQ_\mathrm{f} \times A_\mathrm{f} \to \mathbb{R}$ given by

$$H_{\mathrm{f}\to\mathrm{s}}(x_\mathrm{f}, a_\mathrm{f}) = y_\mathrm{c} - l\cos(\varphi + \theta) - L\sin\theta, \tag{2}$$

signifies the touchdown event at its zero crossing, and defines a smooth switching manifold $S_{\mathrm{f}\to\mathrm{s}}$ in the augmented state space $X_\mathrm{f} = TQ_\mathrm{f} \times A_\mathrm{f}$, given by

$$S_{\mathrm{f}\to\mathrm{s}} = \left\{ (x_\mathrm{f}, a_\mathrm{f}) \in X_\mathrm{f} \,\middle|\, H_{\mathrm{f}\to\mathrm{s}}(x_\mathrm{f}, a_\mathrm{f}) = 0, \dot{y}_\mathrm{c} < 0 \right\}. \tag{3}$$

Note that in (2) and (3), the parameter a_f is available for control, and will eventually be chosen according to an event-based feedback law.

B. Stance Phase Dynamics

The configuration space Q_s of the ASLIP during the stance phase is parameterized by the coordinates $q_\mathrm{s} = (l, \varphi, \theta)' \in Q_\mathrm{s} \subseteq \mathbb{R} \times S^2$. Using the Lagrangian approach and then bringing the equations into standard state-space form, the ASLIP stance dynamics is described by

$$\dot{x}_\mathrm{s} = f_\mathrm{s}(x_\mathrm{s}) + g_\mathrm{s}(x_\mathrm{s})u, \tag{4}$$

where $x_\mathrm{s} \in TQ_\mathrm{s} = \{(q_\mathrm{s}', \dot{q}_\mathrm{s}')' \mid q_\mathrm{s} \in Q_\mathrm{s}, \dot{q}_\mathrm{s} \in \mathbb{R}^3\} = X_\mathrm{s}$ is the state vector, and $u = (u_1, u_2)' \in U \subseteq \mathbb{R}^2$ is the input vector.

The threshold function $H_{\mathrm{s}\to\mathrm{f}} : TQ_\mathrm{s} \times U \to \mathbb{R}$, given by

$$H_{\mathrm{s}\to\mathrm{f}}(x_\mathrm{s}, u) = u_1\cos(\varphi + \theta) - (u_2/l)\sin(\varphi + \theta), \tag{5}$$

specifies the liftoff event at its zero crossing and defines a smooth switching manifold $S_{\mathrm{s}\to\mathrm{f}}$ in the augmented space $TQ_{\mathrm{s}}\times U$, given by

$$S_{\mathrm{s}\to\mathrm{f}}=\left\{(x_{\mathrm{s}},u)\in TQ_{\mathrm{s}}\times U \,\middle|\, H_{\mathrm{s}\to\mathrm{f}}(x_{\mathrm{s}},u)=0\right\}. \qquad (6)$$

Equation (6) describes the fact that liftoff occurs when the vertical component of the ground force, which is a function[1] of the control inputs u_1 and u_2, becomes zero.

C. ASLIP Hybrid Dynamics

Let $\phi_{\mathrm{f}}:[0,\infty)\times X_{\mathrm{f}}\to X_{\mathrm{f}}$ and $\phi_{\mathrm{s}}:[0,\infty)\times X_{\mathrm{s}}\to X_{\mathrm{s}}$ denote the solutions generated by the flight and stance models (1) and (4), respectively. Note that the simplicity of the vector field f_{f} allows for explicit calculation of the flow $\phi_{\mathrm{f}}(t,x_{\mathrm{f}})$. When the flight flow $\phi_{\mathrm{f}}(t,x_{\mathrm{f}})$ intersects $S_{\mathrm{f}\to\mathrm{s}}$, transition from flight to stance occurs. Let $\Delta_{\mathrm{f}\to\mathrm{s}}:S_{\mathrm{f}\to\mathrm{s}}\to X_{\mathrm{s}}$ be the transition map from the flight to the stance phase. Similarly, let $\Delta_{\mathrm{s}\to\mathrm{f}}:S_{\mathrm{s}\to\mathrm{f}}\to X_{\mathrm{f}}$ be the transition map from the stance to the flight phase. Then the open-loop hybrid model of the ASLIP is

$$\Sigma_{\mathrm{f}}:\begin{cases} X_{\mathrm{f}}=TQ_{\mathrm{f}}\times A_{\mathrm{f}} \\ \begin{pmatrix}\dot{x}_{\mathrm{f}}\\ \dot{a}_{\mathrm{f}}\end{pmatrix}=\begin{pmatrix}f_{\mathrm{f}}(x_{\mathrm{f}})\\ 0\end{pmatrix} \\ S_{\mathrm{f}\to\mathrm{s}}=\left\{(x_{\mathrm{f}},a_{\mathrm{f}})\in X_{\mathrm{f}} \,\middle|\, H_{\mathrm{f}\to\mathrm{s}}(x_{\mathrm{f}},a_{\mathrm{f}})=0\right\} \\ x_{\mathrm{s}}^{+}=\Delta_{\mathrm{f}\to\mathrm{s}}(x_{\mathrm{f}}^{-},a_{\mathrm{f}}) \end{cases} \qquad (7)$$

$$\Sigma_{\mathrm{s}}:\begin{cases} X_{\mathrm{s}}=TQ_{\mathrm{s}} \\ \dot{x}_{\mathrm{s}}=f_{\mathrm{s}}(x_{\mathrm{s}})+g_{\mathrm{s}}(x_{\mathrm{s}})u \\ S_{\mathrm{s}\to\mathrm{f}}=\left\{(x_{\mathrm{s}},u)\in TQ_{\mathrm{s}}\times U \,\middle|\, H_{\mathrm{s}\to\mathrm{f}}(x_{\mathrm{s}},u)=0\right\} \\ x_{\mathrm{f}}^{+}=\Delta_{\mathrm{s}\to\mathrm{f}}(x_{\mathrm{s}}^{-},u) \end{cases} \qquad (8)$$

where $x_i^-(t)=\lim_{\tau\uparrow t}x_i(\tau)$ and $x_i^+(t)=\lim_{\tau\downarrow t}x_i(\tau)$, $i\in\{\mathrm{s},\mathrm{f}\}$ are the left and right limits of the stance and flight solutions.

The subsystems Σ_{f} and Σ_{s} can be combined into a single system with impulse effects Σ_{ASLIP} describing the open-loop hybrid dynamics of the ASLIP. Define the time-to-touchdown function $T_{\mathrm{f}}:X_{\mathrm{f}}\to\mathbb{R}\cup\{\infty\}$, as

$$T_{\mathrm{f}}(x_{\mathrm{f},0},a_{\mathrm{f}})=\inf\left\{t\in[0,\infty) \,\middle|\, \phi_{\mathrm{f}}(t,x_{\mathrm{f},0})\in S_{\mathrm{f}\to\mathrm{s}}\right\}. \qquad (9)$$

The flow map[2] $F_{\mathrm{f}}:X_{\mathrm{f}}\to X_{\mathrm{f}}$ for the flight phase can then be given by the rule $(x_{\mathrm{f},0},a_{\mathrm{f}})\mapsto\phi_{\mathrm{f}}\left(T_{\mathrm{f}}(x_{\mathrm{f},0},a_{\mathrm{f}}),x_{\mathrm{f},0}\right)$. Let $\Delta:S_{\mathrm{s}\to\mathrm{f}}\times A_{\mathrm{f}}\to X_{\mathrm{s}}$ be the map

$$\Delta=\Delta_{\mathrm{f}\to\mathrm{s}}\circ\left(F_{\mathrm{f}}\times\mathrm{id}_{A_{\mathrm{f}}}\right)\circ\left(\Delta_{\mathrm{s}\to\mathrm{f}}\times\mathrm{id}_{A_{\mathrm{f}}}\right). \qquad (10)^3$$

where $\mathrm{id}_{A_{\mathrm{f}}}$ is the identity map on A_{f}. The map Δ "compresses" the flight phase into an "event," and can be thought of as a (generalized) "impact map" or "reset map" [5], [8]. In this setting, the hybrid dynamics of the ASLIP take the form

$$\Sigma_{\mathrm{ASLIP}}:\begin{cases} \dot{x}_{\mathrm{s}}=f_{\mathrm{s}}(x_{\mathrm{s}})+g_{\mathrm{s}}(x_{\mathrm{s}})u, & (x_{\mathrm{s}}^{-},u)\notin S_{\mathrm{s}\to\mathrm{f}} \\ x_{\mathrm{s}}^{+}=\Delta(x_{\mathrm{s}}^{-},u,a_{\mathrm{f}}), & (x_{\mathrm{s}}^{-},u,a_{\mathrm{f}})\in S_{\mathrm{s}\to\mathrm{f}}\times A_{\mathrm{f}} \end{cases}. \qquad (11)$$

The left and right limits x_{s}^{-} and x_{s}^{+} correspond to the states "just prior to liftoff" and "just after touchdown" respectively. The system Σ_{ASLIP} is defined on a single chart X_{s}, where the states evolve, together with a the map Δ, which reinitializes the differential equation at liftoff.

III. Target Model: The Energy-Stabilized SLIP

In this section, the target model for our controller is introduced. As was mentioned in the introduction, the purpose of this paper is to introduce a framework for designing controllers of running robots that take advantage of feedback control laws available for the extensively studied SLIP. The standard SLIP consists of a point mass attached to a massless prismatic spring, and it is passive (no torque inputs) and conservative (no energy losses). In this paper, we consider a variant of the SLIP, where the leg force is allowed to be non-conservative. The purpose of this modification is to introduce control authority over the total energy, which, in the standard SLIP, is conserved along solutions, thus precluding the existence of exponentially stable periodic orbits [2], [7]. This system, called the *Energy-Stabilized SLIP (ES-SLIP)*, is presented in Fig. 2.

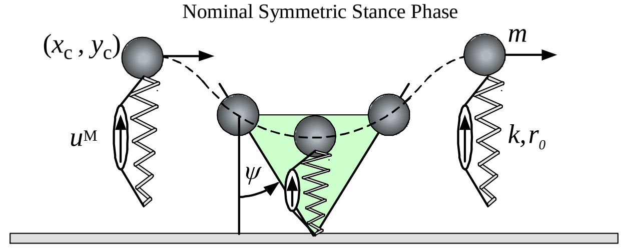

Figure 2. The Energy Stabilized SLIP (ES-SLIP), with an actuator parallel with the spring.

A. ES-SLIP Open-Loop Hybrid Dynamics

The derivation of the hybrid model for the ES-SLIP is similar to that of the ASLIP, thus the exposition in this section will be terse. The flight and stance configuration spaces $Q_{\mathrm{f}}^{\mathrm{M}}$ and $Q_{\mathrm{s}}^{\mathrm{M}}$ respectively will both be parameterized by the Cartesian coordinates of the COM $(x_{\mathrm{c}},y_{\mathrm{c}})\in Q_{\mathrm{f}}^{\mathrm{M}}=Q_{\mathrm{s}}^{\mathrm{M}}=Q^{\mathrm{M}}\subset\mathbb{R}^2$, where the superscript "M"

[1] When a feedback controller $u=\alpha(x_{\mathrm{s}})$ is introduced, the liftoff condition $H_{\mathrm{s}\to\mathrm{f}}(x_{\mathrm{s}},\alpha(x_{\mathrm{s}}))=0$ will only be a function of the states.

[2] The flight flow map presupposes the existence of a time instant t such that $\phi_{\mathrm{f}}(t,x_{\mathrm{f},0})\in S_{\mathrm{f}\to\mathrm{s}}$. The case where such a time instant does not exist does not correspond to periodic running motions.

[3] Notation: let $f_1:X\to X_1$ and $f_2:X\to X_2$, and define $f_1\times f_2:X\to X_1\times X_2$ by $(f_1\times f_2)(x)=(f_1(x),f_2(x))\in X_1\times X_2$, $x\in X$.

denotes the ES-SLIP target model. Hence, the system dynamics evolves in the state space

$$X^{\mathrm{M}} = TQ^{\mathrm{M}} = \left\{ x^{\mathrm{M}} = \mathrm{col}\left(q^{\mathrm{M}}, \dot{q}^{\mathrm{M}}\right) \middle| q^{\mathrm{M}} \in Q^{\mathrm{M}}, \dot{q}^{\mathrm{M}} \in \mathbb{R}^2 \right\}.$$

As in the ASLIP, the ES-SLIP hybrid open-loop dynamics can be written in the following form

$$\Sigma_{\text{ES-SLIP}} : \begin{cases} \dot{x}^{\mathrm{M}} = f_{\mathrm{s}}^{\mathrm{M}}\left(x^{\mathrm{M}}\right) + g_{\mathrm{s}}^{\mathrm{M}}\left(x^{\mathrm{M}}\right)u^{\mathrm{M}}, & \left(\left(x^{\mathrm{M}}\right)^-, u^{\mathrm{M}}\right) \notin S_{\mathrm{s}\to\mathrm{f}}^{\mathrm{M}} \\ \left(x^{\mathrm{M}}\right)^+ = \Delta^{\mathrm{M}}\left(\left(x^{\mathrm{M}}\right)^-, \psi\right), & \left(\left(x^{\mathrm{M}}\right)^-, u^{\mathrm{M}}, \psi\right) \in S_{\mathrm{s}\to\mathrm{f}}^{\mathrm{M}} \times A_{\mathrm{f}}^{\mathrm{M}} \end{cases}, \quad (12)$$

where $u^{\mathrm{M}} \in U^{\mathrm{M}} \subset \mathbb{R}$ is the input and $\psi \in A_{\mathrm{f}}^{\mathrm{M}} \subset S^1$ is the touchdown angle (angle of attack), and $f_{\mathrm{s}}^{\mathrm{M}} : X^{\mathrm{M}} \to TX^{\mathrm{M}}$ and $g_{\mathrm{s}}^{\mathrm{M}} : X^{\mathrm{M}} \to TX^{\mathrm{M}}$ are the system and input vector fields in the stance phase. The switching surface is taken to be the liftoff surface

$$S_{\mathrm{s}\to\mathrm{f}}^{\mathrm{M}} = \left\{ \left(x^{\mathrm{M}}, u^{\mathrm{M}}\right) \in X^{\mathrm{M}} \times U^{\mathrm{M}} \middle| H_{\mathrm{s}\to\mathrm{f}}^{\mathrm{M}}\left(x^{\mathrm{M}}, u^{\mathrm{M}}\right) = 0 \right\}, \quad (13)$$

where $H_{\mathrm{s}\to\mathrm{f}}^{\mathrm{M}} : TQ^{\mathrm{M}} \times U^{\mathrm{M}} \to \mathbb{R}$ is defined as

$$H_{\mathrm{s}\to\mathrm{f}}^{\mathrm{M}}\left(x^{\mathrm{M}}, u^{\mathrm{M}}\right) = \frac{y_{\mathrm{c}}}{\sqrt{x_{\mathrm{c}}^2 + y_{\mathrm{c}}^2}}\left(F_{\mathrm{el}} + u^{\mathrm{M}}\right). \quad (14)$$

In (14), F_{el} is the elastic force developed by the prismatic spring of the leg. Assuming for definiteness that the spring is linear,

$$F_{\mathrm{el}} = k\left(r_0 - \sqrt{x_{\mathrm{c}}^2 + y_{\mathrm{c}}^2}\right), \quad (15)$$

where k is the spring constant and r_0 is the nominal spring length, see Fig. 2. In this work, r_0 is taken to be the uncompressed length of the leg. However, these assumptions can be relaxed to allow for spring pretension and nonlinear spring characteristics.

B. *ES-SLIP Closed-Loop Hybrid Dynamics*

In order to accommodate perturbations away from the nominal energy, the conservative force F_{el} developed by the springy leg of the standard SLIP is modified to include a nonconservative feedback component $u^{\mathrm{M}} = \alpha^{\mathrm{M}}(x^{\mathrm{M}})$. The purpose of u^{M} is to stabilize the total energy of the system at a desired level, and is achieved by

$$\alpha^{\mathrm{M}}\left(x^{\mathrm{M}}\right) = -K_P^E \frac{x_{\mathrm{c}}\dot{x}_{\mathrm{c}} + y_{\mathrm{c}}\dot{y}_{\mathrm{c}}}{\sqrt{x_{\mathrm{c}}^2 + y_{\mathrm{c}}^2}}\left[E\left(x^{\mathrm{M}}\right) - \bar{E}\right], \quad (16)$$

where $\bar{E}$ is the desired nominal energy level, $E(x^{\mathrm{M}})$ is the total energy of the SLIP, and K_P^E is a positive gain.

To regulate the forward speed, the following event-based control law is employed

$$\psi\left(\dot{x}_{\mathrm{c}}^-\right) = \bar{\psi} + K_{\dot{x}}\left(\dot{x}_{\mathrm{c}}^- - \bar{\dot{x}}_{\mathrm{c}}\right), \quad (17)$$

where $\bar{\psi}$ and $\bar{\dot{x}}_{\mathrm{c}}$ specify the nominal touchdown angle and forward speed respectively, $\dot{x}_{\mathrm{c}}^-$ is the actual forward speed just prior to liftoff, and $K_{\dot{x}}$ is a positive gain. It can be recognized that (17) corresponds to a variation of Raibert's speed controller, [12].

Substituting the feedback laws (16) and (17) in (12), the closed-loop ES-SLIP hybrid dynamics can be obtained as

$$\Sigma_{\text{ES-SLIP}}^{\mathrm{cl}} : \begin{cases} \dot{x}^{\mathrm{M}} = f_{\mathrm{s,cl}}^{\mathrm{M}}\left(x^{\mathrm{M}}\right), & \left(x^{\mathrm{M}}\right)^- \notin \hat{S}_{\mathrm{s}\to\mathrm{f}}^{\mathrm{M}} \\ \left(x^{\mathrm{M}}\right)^+ = \Delta_{\mathrm{cl}}^{\mathrm{M}}\left(\left(x^{\mathrm{M}}\right)^-\right), & \left(x^{\mathrm{M}}\right)^- \in \hat{S}_{\mathrm{s}\to\mathrm{f}}^{\mathrm{M}} \end{cases}, \quad (18)$$

where,

$$\hat{S}_{\mathrm{s}\to\mathrm{f}}^{\mathrm{M}} = \left\{ x^{\mathrm{M}} \in X^{\mathrm{M}} \middle| H_{\mathrm{s}\to\mathrm{f}}^{\mathrm{M}}\left(x^{\mathrm{M}}, \alpha^{\mathrm{M}}\left(x^{\mathrm{M}}\right)\right) = 0 \right\}, \quad (19)$$

and $f_{\mathrm{s,cl}}^{\mathrm{M}}\left(x^{\mathrm{M}}\right) = f_{\mathrm{s}}^{\mathrm{M}}\left(x^{\mathrm{M}}\right) + g_{\mathrm{s}}^{\mathrm{M}}\left(x^{\mathrm{M}}\right)\alpha^{\mathrm{M}}\left(x^{\mathrm{M}}\right)$ is the closed-loop stance vector field, which is given below for future use,

$$f_{\mathrm{s,cl}}^{\mathrm{M}}\left(x^{\mathrm{M}}\right) = \begin{pmatrix} \dot{x}_{\mathrm{c}} \\ \dot{y}_{\mathrm{c}} \\ \frac{x_{\mathrm{c}}}{\sqrt{x_{\mathrm{c}}^2 + y_{\mathrm{c}}^2}}\left(F_{\mathrm{el}} + \alpha^{\mathrm{M}}\left(x^{\mathrm{M}}\right)\right) \\ \frac{y_{\mathrm{c}}}{\sqrt{x_{\mathrm{c}}^2 + y_{\mathrm{c}}^2}}\left(F_{\mathrm{el}} + \alpha^{\mathrm{M}}\left(x^{\mathrm{M}}\right)\right) - mg \end{pmatrix}. \quad (20)$$

In order to study the stability properties of periodic orbits of $\Sigma_{\text{ES-SLIP}}^{\mathrm{cl}}$, the method of Poincaré will be used. The Poincaré section is selected to be the surface $\hat{S}_{\mathrm{s}\to\mathrm{f}}^{\mathrm{M}}$ defined by (19). Let $\phi_{\mathrm{s,cl}}^{\mathrm{M}} : [0,\infty) \times X^{\mathrm{M}} \to X^{\mathrm{M}}$ be the flow generated by $f_{\mathrm{s,cl}}^{\mathrm{M}}$, and define the time-to-liftoff function $T_{\mathrm{s}}^{\mathrm{M}} : X^{\mathrm{M}} \to \mathbb{R} \cup \{\infty\}$, in a similar fashion as (9), by

$$T_{\mathrm{s}}^{\mathrm{M}}\left(x_{\mathrm{s},0}^{\mathrm{M}}\right) = \inf\left\{ t \in [0,\infty) \middle| \phi_{\mathrm{s,cl}}^{\mathrm{M}}\left(t, x_{\mathrm{s},0}^{\mathrm{M}}\right) \in \hat{S}_{\mathrm{s}\to\mathrm{f}}^{\mathrm{M}} \right\}. \quad (21)$$

Then, the Poincaré map $P^{\mathrm{M}} : \hat{S}_{\mathrm{s}\to\mathrm{f}}^{\mathrm{M}} \to \hat{S}_{\mathrm{s}\to\mathrm{f}}^{\mathrm{M}}$ is given by

$$P^{\mathrm{M}}\left(x^{\mathrm{M}}\right) = \phi_{\mathrm{s,cl}}^{\mathrm{M}}\left(T_{\mathrm{s}}^{\mathrm{M}} \circ \Delta_{\mathrm{cl}}^{\mathrm{M}}\left(x^{\mathrm{M}}\right), \Delta_{\mathrm{cl}}^{\mathrm{M}}\left(x^{\mathrm{M}}\right)\right). \quad (22)$$

Note that feedback control laws similar to (16) and (17) exist in the literature, and the particular ones used here are for illustrative purposes only. It is emphasized that *any* other in-stride or event-based controller could have been used to stabilize the SLIP. For instance, energy stabilization in nonconservative monopedal models has been demonstrated using linear (leg) and rotational (hip) actuation in [1] and [4], respectively. On the other hand, a large variety of event-based controllers exist for the SLIP e.g. [2], [12], [14], [16], which are known to have very appealing properties such as a large domain of attraction. In this work, we develop formally a controller for the ASLIP that affords the direct use of control laws available for the SLIP.

IV. Main Theorem: Controller Design

The control action takes place in two hierarchical levels. In the first level, continuous in-stride control is exerted during the stance phase to stabilize the torso at a desired posture and to create an invariant manifold on which the ES-SLIP dynamics can be imposed. In the second level, an event-based SLIP controller is used to stabilize a periodic orbit of the system. These results are summarized in the following theorem and corollary.

Theorem 1: Hybrid controller

Let $\tilde{Q}_{\mathrm{s}} = \{q_{\mathrm{s}} \in Q_{\mathrm{s}} \mid l \neq L\sin\varphi\}$. Then, there exists a map $\Phi : T\tilde{Q}_{\mathrm{s}} \to \mathbb{R}^6$ that is a diffeomorphism onto its image, and such that, in coordinates $\Phi(x_s) = x = (\eta', z')' \in \mathbb{R}^6$, the following hold:

A. In-stride Continuous Control

For every $\varepsilon > 0$, there exists a C^1 feedback control law $u = \alpha(x_{\mathrm{s}}, \varepsilon)$, such that the model

$$f_{\mathrm{s,cl}}(x_{\mathrm{s}}, \varepsilon) = f_{\mathrm{s}}(x_{\mathrm{s}}) + g_{\mathrm{s}}(x_{\mathrm{s}})\alpha(x_{\mathrm{s}}, \varepsilon), \tag{23}$$

satisfies:

A.1) the vector field

$$\tilde{f}_{\mathrm{s,cl}}(x, \varepsilon) = \left(\frac{\partial \Phi}{\partial x_{\mathrm{s}}} f_{\mathrm{s,cl}}(x_{\mathrm{s}}, \varepsilon)\right)_{x_{\mathrm{s}} = \Phi^{-1}(x)} \tag{24}$$

has the form

$$\tilde{f}_{\mathrm{s,cl}}(x, \varepsilon) = \begin{pmatrix} \tilde{f}_{\mathrm{s,cl,1:2}}(\eta, \varepsilon) \\ \tilde{f}_{\mathrm{s,cl,3:6}}(z, \eta) \end{pmatrix}; \tag{25}$$

A.2) the set $Z = \{x \in T\tilde{Q}_{\mathrm{s}} \mid \eta = 0\}$ is a smooth 4-dimensional C^1 embedded submanifold of $\mathbb{R}^6$ and is invariant under the stance flow, i.e. $x \in Z$ implies $\tilde{f}_{\mathrm{s,cl}}(x, \varepsilon) \in T_x Z$;

A.3) the restriction dynamics

$$\tilde{f}_{\mathrm{s,cl}}(x, \varepsilon)\big|_Z = \tilde{f}_{\mathrm{s,cl,3:6}}(z, 0) \tag{26}$$

is diffeomorphic to the ES-SLIP stance phase closed-loop dynamics $f^{\mathrm{M}}_{\mathrm{s,cl}}(x^{\mathrm{M}})$ given by (20).

B. Exponentially Contracting Transverse Dynamics

B.1) $\tilde{f}_{\mathrm{s,cl,1:2}}(\eta, \varepsilon)$ takes the form

$$\tilde{f}_{\mathrm{s,cl,1:2}}(\eta, \varepsilon) = A(\varepsilon)\eta, \tag{27}$$

and $\lim_{\varepsilon \searrow 0} e^{A(\varepsilon)} = 0$.

C. Event-based control

There exists a C^1 event-based control law $a_{\mathrm{f}} = \beta(x^-)$ such that the map

$$\Delta_{\mathrm{cl}}(x^-) = \left[\Delta \circ \left(\Phi^{-1} \times (\alpha \circ \Phi^{-1}) \times \beta\right)\right](x^-) \tag{28}$$

satisfies:

C.1) $S_{\mathrm{s}\to\mathrm{f}} \cap Z$ is a smooth co-dimension one submanifold of Z and $\Delta_{\mathrm{cl}}(S_{\mathrm{s}\to\mathrm{f}} \cap Z) \subset Z$;

C.2) the restricted reset map $\Delta_{\mathrm{cl}}(x)|_Z$ is diffeomorphic to the ES-SLIP closed-loop reset map $\Delta^{\mathrm{M}}_{\mathrm{cl}}$.■

The proof of Theorem 1 will be given in Section V.

For $\varepsilon > 0$ a given constant, the closed-loop hybrid dynamics of the ASLIP under the continuous and event-based feedback control laws of Theorem 1 takes the form

$$\Sigma^{\mathrm{cl}}_{\mathrm{ASLIP}} : \begin{cases} \dot{x} = \tilde{f}_{\mathrm{s,cl}}(x, \varepsilon), & x^- \notin \hat{S}_{\mathrm{s}\to\mathrm{f}} \\ x^+ = \Delta_{\mathrm{cl}}(x^-), & x^- \in \hat{S}_{\mathrm{s}\to\mathrm{f}} \end{cases}, \tag{29}$$

where

$$\hat{S}_{\mathrm{s}\to\mathrm{f}} = \left\{x \in T\tilde{Q}_{\mathrm{s}} \,\middle|\, H_{\mathrm{s}\to\mathrm{f}}\left(x, \alpha\left(\Phi^{-1}(x), \varepsilon\right)\right) = 0\right\}. \tag{30}$$

The stability properties of $\Sigma^{\mathrm{cl}}_{\mathrm{ASLIP}}$ will be studied via the corresponding Poincaré return map. As in Section III, let $\phi_{\mathrm{s,cl}} : [0, \infty) \times X_{\mathrm{s}} \to X_{\mathrm{s}}$ be the flow generated by $\tilde{f}_{\mathrm{s,cl}}$, and $T_{\mathrm{s}} : X_{\mathrm{s}} \to \mathbb{R} \cup \{\infty\}$ be the time-to-liftoff function defined as

$$T_{\mathrm{s}}(x_0) = \inf\left\{t \in [0, \infty) \,\middle|\, \phi_{\mathrm{s,cl}}(t, x_0) \in \hat{S}_{\mathrm{s}\to\mathrm{f}}\right\}. \tag{31}$$

Then, the Poincaré return map $P : \hat{S}_{\mathrm{s}\to\mathrm{f}} \to \hat{S}_{\mathrm{s}\to\mathrm{f}}$ is given by

$$P(x) = \phi_{\mathrm{s,cl}}\left(T_{\mathrm{s}} \circ \Delta_{\mathrm{cl}}(x), \Delta_{\mathrm{cl}}(x)\right). \tag{32}$$

The following Corollary 1 is an immediate consequence of Theorem 1 in view of the results in [9].

Corollary 1: Exponential stability of $\Sigma^{\mathrm{cl}}_{\mathrm{ASLIP}}$

Let $(x^{\mathrm{M}})^*$ be a fixed point of P^{M} and x^* a fixed point of P. There exist $\bar{\varepsilon} > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon})$, x^* is exponentially stable, if, and only if, $(x^{\mathrm{M}})^*$ is exponentially stable.■

Remark 1. The intuitive meaning of Corollary 1 is that, for given controllers that create an exponentially stable periodic orbit of the ES-SLIP, the feedback laws $u = \alpha(x_{\mathrm{s}}, \varepsilon)$ and $a_{\mathrm{f}} = \beta(x^-)$ specified in Theorem 1 render this orbit exponentially stable in the ASLIP.

V. Proof of The Main Theorem

In this section, Theorem 1 is proved through a sequence of Lemmas. The procedure is constructive, and results in a control law satisfying the requirements of Theorem 1.

A. In-stride Continuous Control

The purpose of the in-stride control action during the stance phase is twofold. First, it ensures that the torso remains at a desired (constant and upright) pitch angle, and second, it renders the translational dynamics of the ASLIP diffeomorphic to the ES-SLIP closed-loop stance dynamics. This prepares the continuous part of $\Sigma^{\mathrm{cl}}_{\mathrm{ASLIP}}$ in (29) so that any event-based controller that exponentially stabilizes a periodic orbit of the SLIP can be used to achieve exponential stability of the ASLIP orbit. In view of the underactuated nature of the stance phase, the two control objectives will be achieved in different time scales. Since the requirement for the torso being upright throughout the motion is more stringent, high-gain control will be imposed on the pitch rotational motion. Hence, the system will be decomposed into fast and slow dynamics governing the rotational and the translational dynamics of the torso, respectively.

The continuous part of Σ_{ASLIP} can be written as

$$\dot{x}_s = f_s(x_s) + g_{s,1}(x_s)u_1 + g_{s,2}(x_s)u_2 . \tag{33}$$

Define the output function $h : \tilde{Q}_s \to \mathbb{R}$ by

$$y = h(q_s) = \theta - \bar{\theta} , \tag{34}$$

where $\bar{\theta}$ is a desired pitch angle, taken to be a constant. It can formally be shown that $\bar{\theta}$ being constant is a *necessary* condition for the existence of an embedding control law. Due to limited space, the proof of this statement will not be presented here. The output defined by (34) results in the second-order input-output dynamics

$$\frac{d^2 y}{dt^2} = \left[L_{f_s}^2 h(x_s) + L_{g_{s,1}} L_{f_s} h(q_s) u_1 \right] + L_{g_{s,2}} L_{f_s} h(q_s) u_2 , \tag{35}$$

where

$$L_{f_s}^2 h(x_s) = 0 ,$$

$$L_{g_{s,1}} L_{f_s} h(q_s) = \frac{-L\cos\varphi}{J} ,\ L_{g_{s,2}} L_{f_s} h(q_s) = \frac{L\sin\varphi - l}{Jl} . \tag{36}$$

Lemma 1: Stance phase zero dynamics

Under the output function h defined by (34), and for $q_s \in \tilde{Q}_s = \{q_s \in Q_s \mid l \neq L\sin\varphi\}$,

1) the set $Z = \{x_s \in T\tilde{Q}_s \mid h(x_s) = 0, L_{f_s} h(x_s) = 0\}$ is a smooth 4-dimensional submanifold of $T\tilde{Q}_s$;

2) the feedback control law

$$u_2^* = -\frac{L_{g_{s,1}} L_{f_s} h(x_s)}{L_{g_{s,2}} L_{f_s} h(x_s)} u_1 \tag{37}$$

renders Z invariant under the stance dynamics; that is, for all $x_s \in Z$, $u_1 \in \mathbb{R}$,

$$f_s(x_s) + g_{s,1}(x_s)u_1 + g_{s,2}(x_s)u_2^* \in T_{x_s} Z ;$$

3) there exist smooth functions $\gamma_1(x_s)$ and $\gamma_2(x_s)$ so that the map $\Phi : T\tilde{Q}_s \to \mathbb{R}^6$

$$\Phi(x_s) = x = (\eta_1, \eta_2, z_1', z_2')' , \tag{38}$$

where

$$\eta_1 = h(q_s) ,\ \eta_2 = L_{f_s} h(x_s) ,$$

$$z_1 = (l, \varphi)' , z_2 = (\gamma_1(x_s), \gamma_2(x_s))' , \tag{39}$$

is a valid coordinate transformation, i.e. Φ is a diffeomorphism onto its image, and

$$L_{g_{s,2}} \gamma_1(x_s) = 0 ,\ L_{g_{s,2}} \gamma_2(x_s) = 0 .\blacksquare \tag{40}$$

Proof:

Parts 1) and 2) of Lemma 1 follow from general results in [10]. For part 3), consider the distribution $G = \text{span}\{g_{s,2}\}$, which has constant dimension $d = 1$ on $T\tilde{Q}_s$. Since G is one dimensional, it is involutive and thus, by the Frobenius theorem, integrable. As a result there exist $n - d = 6 - 1 = 5$ real-valued functions defined on $T\tilde{Q}_s$ such that the annihilator $G^\perp$ of G is $G^\perp = \text{span}\{dl, d\varphi, d\theta, d\gamma_1, d\gamma_2\}$. A straightforward application of the constructive proof of the sufficiency part of Frobenius theorem [10] results in

$$\gamma_1(x_s) = \dot{l} + L\dot{\theta}\cos\varphi , \tag{41}$$

$$\gamma_2(x_s) = \dot{\varphi} + \left[-1 + \frac{L\sin\varphi}{l} + \frac{J}{ml(L\sin\varphi - l)} \right] \dot{\theta} . \tag{42}$$

Finally, it is straightforward to check that Φ is a diffeomorphism onto its image in $\mathbb{R}^6$.$\blacksquare$

Let $\varepsilon > 0$ and define the feedback

$$u_2 = \alpha_2(x_s, \varepsilon) = \frac{1}{L_{g_{s,2}} L_{f_s} h(x_s)} \left(\upsilon(\theta, \dot{\theta}, \varepsilon) - L_{g_{s,1}} L_{f_s} h(x_s) u_1 \right) , \tag{43}$$

where

$$\upsilon(\theta, \dot{\theta}, \varepsilon) = -\frac{1}{\varepsilon^2} K_P^\theta (\theta - \bar{\theta}) - \frac{1}{\varepsilon} K_V^\theta \dot{\theta} , \tag{44}$$

and K_P^θ, K_V^θ are positive constants. Under this feedback law, the model (33) becomes

$$\dot{x}_s = \tilde{f}_s(x_s, \varepsilon) + \tilde{g}_s(x_s) u_1 , \tag{45}$$

where

$$\tilde{f}_s(x_s, \varepsilon) = f_s(x_s) + \left[\frac{1}{L_{g_{s,2}} L_{f_s} h(x_s)} \upsilon(\theta, \dot{\theta}, \varepsilon) \right] g_{s,2}(x_s) , \tag{46}$$

$$\tilde{g}_s(x_s) = \left[g_{s,1}(x_s) - g_{s,2}(x_s) \frac{L_{g_{s,1}} L_{f_s} h(x_s)}{L_{g_{s,2}} L_{f_s} h(x_s)} \right] . \tag{47}$$

Under the coordinates of Lemma 1, (45) has the form

$$\dot{\eta} = A(\varepsilon)\eta , \tag{48}$$

$$\dot{z} = f_z(\eta, z) + g_z(z) u_1 . \tag{49}$$

With the additional change of coordinates $\tilde{\eta}_1 = \eta_1 / \varepsilon$ and $\tilde{\eta}_2 = \eta_2$, the model (48)-(49) can be written as

$$\varepsilon \dot{\tilde{\eta}} = \tilde{A} \tilde{\eta} , \tag{50}$$

$$\dot{z} = f_z(\tilde{\eta}, z, \varepsilon) + g_z(z) u_1 , \tag{51}$$

where

$$\tilde{A} = \begin{pmatrix} 0 & 1 \\ -K_p^\theta & -K_v^\theta \end{pmatrix} . \tag{52}$$

Setting $\varepsilon = 0$, (50) reduces to the algebraic equation

$\tilde{A}\tilde{\eta}=0$, which, by properly selecting the gains $\{K_p^\theta, K_v^\theta\}$ in (52), has the origin as its unique solution. Hence, (50)-(51) is in standard singular perturbation form and the corresponding reduced model is obtained by substituting $\varepsilon=0$ and $\eta=0$ in the slow part of the dynamics (51), i.e.

$$\dot{z}=f_z(0,z,0)+g_z(z)u_1, \tag{53}$$

where direct calculation leads to

$$f_z(z)=\begin{pmatrix} z_3 \\ z_4 \\ z_1 z_4^2 - g\cos(\bar{\theta}+z_2) \\ \dfrac{-2z_3 z_4 + g\sin(\bar{\theta}+z_2)}{z_1} \end{pmatrix},\ g_z(z)=\begin{pmatrix} 0 \\ 0 \\ 1/m \\ \dfrac{L\cos z_2}{m z_1(L\sin z_2 - z_1)} \end{pmatrix}. \tag{54}$$

Lemma 2: Restriction dynamics

If $\bar{\theta}$ is the desired pitch angle in (34), define

$$r(z)=\sqrt{z_1^2+L^2-2Lz_1\sin z_2}\,, \tag{55}$$

$$\dot{r}(z)=\frac{z_1-L\sin z_2}{r(z)}z_3-\frac{Lz_1\cos z_2}{r(z)}z_4, \tag{56}$$

$$y(z)=z_1\cos(z_2+\bar{\theta})+L\sin\bar{\theta}\,. \tag{57}$$

Then, if $\bar{E}$ is the desired energy level, the feedback law

$$u_1=\tilde{\alpha}_1(z)=\frac{z_1-L\sin z_2}{r(z)}F_{\text{ES-SLIP}}(z), \tag{58}$$

with

$$F_{\text{ES-SLIP}}(z)=k\left[r_0-r(z)\right]-K_P^E\dot{r}(z)\left[E(z)-\bar{E}\right], \tag{59}$$

$$E(z)=\frac{1}{2}m\left(z_3^2+z_1^2z_4^2\right)+mg\,y(z)+\frac{1}{2}k\left[r_0-r(z)\right]^2, \tag{60}$$

and $K_P^E>0$, renders the restriction dynamics (26) diffeomorphic to the SLIP closed-loop dynamics $f_{\text{s,cl}}^{\text{M}}(x^{\text{M}})$.

Proof:

Substitution of (58) into (53) gives

$$\dot{z}=f_z(z)+g_z(z)\tilde{\alpha}_1(z)=f_{z,\text{cl}}(z). \tag{61}$$

Define the map $\Phi_z: Z\to\mathbb{R}^4$ by

$$\Phi_z(z)=\begin{pmatrix} -z_1\sin(z_2+\bar{\theta})+L\cos\bar{\theta} \\ z_1\cos(z_2+\bar{\theta})+L\sin\bar{\theta} \\ -z_3\sin(z_2+\bar{\theta})-z_1z_4\cos(z_2+\bar{\theta}) \\ z_3\cos(z_2+\bar{\theta})-z_1z_4\sin(z_2+\bar{\theta}) \end{pmatrix}. \tag{62}$$

It is straightforward to check that Φ_z is a diffeomorphism onto its image, thus it describes a valid coordinate transformation on Z. Observe that $\Phi_z(z)=x^{\text{M}}$. The result

$$\left(\frac{\partial\Phi_z}{\partial z}f_{z,\text{cl}}(z)\right)_{x^{\text{M}}=\Phi_z^{-1}(z)}=f_{\text{s,cl}}^{\text{M}}(x^{\text{M}}) \tag{63}$$

is obtained after straightforward algebraic manipulations.■

B. *Stride-to-stride Control*

The purpose of the stride-to-stride controller is to arrange the configuration of the ASLIP at liftoff so that the manifold $S_{\text{s}\to\text{f}}\cap Z$ is invariant under the reset map Δ_{cl}.

Lemma 3: Event-based controller

Let $\bar{\dot{x}}_{\text{c}}$ and $\bar{\psi}$ be the forward running speed at liftoff and the touchdown angle, respectively, corresponding to a (desired) fixed point of the ES-SLIP. Define

$$\psi(x_{\text{s}}^-)=\bar{\psi}+K_{\dot{x}}\left(\dot{x}_{\text{c}}^-(x_{\text{s}}^-)-\bar{\dot{x}}_{\text{c}}\right), \tag{64}$$

where $\dot{x}_{\text{c}}^-$ is the forward running speed of the ASLIP prior to liftoff. Then, the controller $a_{\text{f}}=\beta\left(\Phi(x_{\text{s}}^-)\right)=(l(x_{\text{s}}^-),\varphi(x_{\text{s}}^-))'$,

$$l(x_{\text{s}}^-)=\sqrt{L^2+r_0^2+2Lr_0\sin\left(\psi(x_{\text{s}}^-)-\bar{\theta}\right)}\,, \tag{65}$$

$$\varphi(x_{\text{s}}^-)=\operatorname{asin}\left[\frac{l^2(x_{\text{s}}^-)+L^2-r_0^2}{2Ll(x_{\text{s}}^-)}\right], \tag{66}$$

where $\bar{\theta}$ is the desired pitch angle in (34), achieves C.1) and C.2) of Theorem 1.■

Proof:

Suppose $x\in\hat{S}_{\text{s}\to\text{f}}\cap Z$. To show C.1) notice that this implies $\dot{\theta}^-=0$ and $\theta^-=\bar{\theta}$ just prior to liftoff. Since during the flight phase $\ddot{\theta}=0$, i.e. $\theta(t)\equiv\bar{\theta}$, at touchdown, we have $\dot{\theta}^+=0$ and $\theta^+=\bar{\theta}$, which means that $x^+\in Z$. This establishes hybrid invariance i.e. $\Delta_{\text{cl}}(\hat{S}_{\text{s}\to\text{f}}\cap Z)\subset Z$. The rest of the proof is a consequence of the fact that the flight flow of the ES-SLIP is the same as the translational part of the flight flow of the ASLIP. Equations (65) and (66) ensure that, not only the flight flows, but also the corresponding reset maps, are identical.■

C. *Proof of Theorem 1*

The proof of Theorem 1 follows from Lemmas 1, 2 and 3.

VI. Simulation Results

This section presents a simulation of the controller described above. The mechanical properties of the ASLIP correspond to preliminary designs of a biped robot that is currently under construction, and are given in Table I (see Fig. 1). The desired pitch angle $\bar{\theta}$, the nominal leg length l_0 and the touchdown angle $\bar{\varphi}_{\text{td}}$ are specified a priori, based on gait requirements and design constraints. Then the nominal leg length of the ES-SLIP is calculated through

$$r_0=\sqrt{l_0^2+L^2-2l_0L\sin\bar{\varphi}_{\text{td}}}\,. \tag{67}$$

The ES-SLIP mass coincides with the ASLIP mass while the spring constant k is arbitrarily specified. Table I presents the parameter values used in the simulations.

Given these parameters, an exponentially stable fixed point for the Poincaré return map of the ES-SLIP is calculated. The fixed point is chosen so that it satisfies the desired specifications (e.g. forward running speed), and is then achieved on the ASLIP using the controller described in Theorem 1. In the results presented here, the gains are $K_P^{\theta} = 300$, $K_V^{\theta} = 30$, $\varepsilon = 1.2$, $K_P^{E} = 2$ and $K_{\dot{x}} = 0.2$. Figure 3 presents ASLIP variables as it recovers from a perturbation $\delta\theta = -10\,\text{deg}$ and $\delta\dot{x} = 1\,\text{m/s}$. These values were chosen to highlight the performance of the controller. Indeed, as is shown in Figure 3, the large domain of attraction of Raibert's controller for the SLIP is inherited in the ASLIP, while the in-stride controller rejects perturbations in the pitch angle and fixes the total energy to its nominal value. The ability to take advantage of existing high performance controllers of the SLIP is one of the benefits of this method.

TABLE I

SIMULATION PARAMETERS

Parameter	Value	Units
Torso Mass (m)	27	kg
Torso Inertia (J)	1	kg m^2
Hip to COM Spacing (L)	0.25	m
Nominal ASLIP Leg Length (l_0)	0.9	m
Nominal ASLIP Leg Angle ($\bar{\varphi}_{td}$)	-60	deg
Desired ASLIP Pitch angle ($\bar{\theta}$)	80	deg
ES-SLIP Spring Constant (k)	7600	N/m

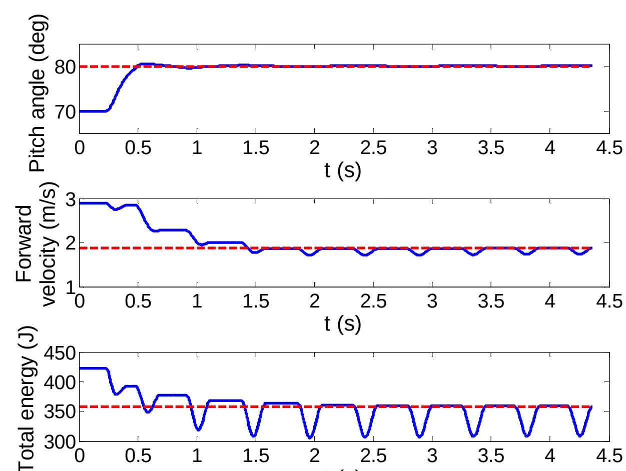

Fig. 3. Pitch angle, COM forward velocity, and energy of the ASLIP. The dashed lines show desired nominal (fixed point) values.

VII. CONCLUSION

In this paper, a framework for the systematic design of control laws with provable properties for the ASLIP, an extension of the SLIP that includes nontrivial torso pitch dynamics, is proposed. The ASLIP can be envisioned as a "building block" toward the construction of controllers for more elaborate models that constitute more accurate representations of legged robots. The control law proposed in this paper acts on two levels. In the first level, continuous in-stride control asymptotically stabilizes the torso pitch and creates an invariant surface on which the closed-loop ASLIP dynamics is diffeomorphic to the target SLIP dynamics. In the second level, an event-based SLIP controller is used to stabilize the system along a desired periodic orbit. An immediate practical consequence of this method is that it affords the direct use of a large body of controller results that are available in the literature for the SLIP. Elsewhere, implementation issues will be addressed and the energetic benefits of this approach will be demonstrated.

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