Controlled Invariance for Discrete-Time Nonlinear Systems with an Application to the Disturbance Decoupling Problem

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Abstract—Invariant distributions are defined for discrete-time nonlinear control systems, and necessary and sufficient conditions are given for their controlled invariance. This extends to discrete-time systems the basic tool which has been so important in solving the various synthesis problems for continuous-time systems. To indicate their utility in the discrete-time setting, they are used to locally solve the disturbance decoupling problem.

I. INTRODUCTION

IN Wonham's now classic text on multivariable control theory [1], a geometric approach is expounded for analyzing the structure of continuous-time linear systems, and for solving important synthesis problems such as disturbance decoupling and noninteracting control. The fundamental notions employed are those of invariant subspaces, (A, B) invariant subspaces, and controllability subspaces. For a linear system $\dot{x} = Ax + Bu$, a subspace V is said to be invariant if $AV \subset V$, and (A, B) invariant if $AV \subset V + \text{Im}B$. The crucial result relating the two concepts is that a subspace can be made invariant through the use of feedback if, and only if, it is (A, B) invariant. Of course, these notions remain equally valid for discrete-time linear systems, $x_{k+1} = Ax_k + Bu_k$.

For the class of continuous-time nonlinear control systems, through the use of differential geometric techniques, a completely parallel theory has been developed. In this theory, a nonlinear system $\dot{x} = f(x, u)$ is regarded as a vector field, parameterized by the control u, defined on some manifold M. The generalization of an invariant subspace is an invariant distribution. Roughly speaking, a distribution is a collection of vector fields [2] which is closed (pointwise) under vector addition and scalar multiplication. Given an affine nonlinear system $\dot{x} = f_o(x) + \sum_{i=1}^{m} u_i f_i(x)$, a distribution Δ is invariant if $[f_i, X] \in \Delta$, for $i = 0, \cdots, m$, for all $X \in \Delta$, where $[\cdot, \cdot]$ denotes the Lie bracket of two vector fields [2]. It is called (f, g) [3], [4] (or A-B [5]) invariant if $[f_i, X] \subset \Delta + \text{span } \{f_1, \cdots, f_m\}$ for $i = 0, \cdots, m$ and for every $X \in \Delta$. The analogy with the linear case should be clear. Here also, given certain regularity hypotheses, one has the important result that a distribution Δ can be made invariant through the use of feedback if, and only if, Δ is (f, g) invariant. Using these ideas, plus a nonlinear generalization of the controllability subspaces [6], [7], necessary and sufficient conditions have been given for the local solvability of the disturbance decoupling [3], [5], [8]–[10], noninteracting control [3], [10]–[13], and invertibility [14]–[16] problems. The award winning paper [3] gives a particularly excellent account of these methods and their application.

Despite the remarkable success of the geometric approach in extending to continuous-time nonlinear systems many important results only previously known for linear systems, nothing of the kind can be said to have occurred for their discrete-time

counterparts. Given the pervasiveness of digital techniques in control applications, it would seem to be especially important, and useful, to extend the geometric theory to the class of discrete-time nonlinear control systems. The goal of this paper is to take the first step in this direction. A theory of invariant distributions will be developed which completely parallels the corresponding theory in continuous time. In particular, necessary and sufficient conditions will be given for their local controlled invariance, and an algorithm will be given for calculating the maximal locally controlled invariant distribution contained in a given involutive distribution. To show that these results are just as important in the discrete setting as in the continuous one, they will be used to locally solve the disturbance decoupling problem.

That the above can be carried out for discrete-time systems, where there is a "lack" of vector fields, and hence Lie brackets, may at first be surprising. However, it should be noted that in [17], [18], where controlled invariance is discussed for *general* nonlinear continuous-time systems, for the first time ever, the necessary and sufficient conditions for controlled invariance were stated without using Lie brackets. It is basically this fact, along with some other concepts introduced in these papers, which permits the extension to the discrete-time case. Indeed, the proofs of the main results of this paper will parallel to a large extent the proofs of the corresponding results given in [17], [18], although in fact, they will be a little easier as it will not be necessary to deal with prolongations of vector fields.

For reasons of preciseness of notation and generality,[1] this paper will employ the language of differential geometry. Some good engineering references for this material are [19]–[25]; standard mathematical texts are [2], [26]–[30]. The reader simply wishing to know what the results are for discrete-time systems on $\mathbb{R}^n$ expressed in rectangular (or other globally defined) coordinates will find this material in the numerous "examples" which are dispersed throughout the paper. Indeed, this is the main purpose of the examples, as realistic applications of these results will have to be the subject of a separate investigation.

The rest of the paper is organized as follows. In Section II, invariant distributions are introduced, and necessary and sufficient conditions are given for their local controlled invariance. In Section III, maximal locally controlled invariant distributions will be shown to exist, and an algorithm will be given for their calculation. In Section IV a disturbance decoupling problem will be formulated and then solved using the results of the previous two sections. Section V contains the conclusions and comments. In the Appendix, the disturbance decoupling problem with partial measurements of the disturbances is considered.

II. CONTROLLED INVARIANCE

This section will fix the notation employed for a nonlinear discrete-time system and introduce the notion of an invariant

[1] For many (continuous-time) nonlinear systems, the natural state space is not a vector space. In view of the importance of sampled-data systems, this is also true in discrete time.

equivalence relation. It will be shown that for those equivalence relations arising from foliations, an infinitesimal necessary and sufficient condition can be given for their local controlled invariance. This will entail extending to discrete-time systems the notion of a controlled invariant distribution, which previously has only been used in studying structural properties of continuous-time systems, and which is a generalization of the (A, B) invariant subspaces so important in geometric linear system theory.

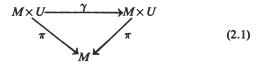The following coordinate-free definition encompasses a large class of discrete-time systems.

Definition 2.1: A nonlinear discrete-time system is a 3-tuple $\Sigma(M \times U, M, f)$ where M and U are analytic manifolds and $f{:}M \times U \to M$ is an analytic function. The points of M are the state space and those of U are the inputs. The system's dynamics are defined by $x_{k+1} = f(x_k, u_k)$.

Example 2.1:

a) Let $M = \mathbb{R}^n$, $U = \mathbb{R}^m$ and let $f{:}\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be any analytic function. Then $\Sigma(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n, f)$ gives the usual class of discrete-time systems on $\mathbb{R}^n$.

b) Taking $f(x, u) = Ax + Bu$, for $A{:}\mathbb{R}^n \to \mathbb{R}^n$ and $B{:}\mathbb{R}^m \to \mathbb{R}^n$ linear operators, gives the ubiquitous discrete-time linear system $x_{k+1} = Ax_k + Bu_k$ on $\mathbb{R}^n$.

Definition 2.2: A *feedback* function γ is an analytic diffeomorphism (i.e., an analytic function which has an analytic inverse) such that the following diagram commutes:

$$
\begin{array}{ccc}
M \times U & \xrightarrow{\quad \gamma \quad} & M \times U \\
 & \searrow{\scriptstyle \pi} \qquad {\scriptstyle \pi}\swarrow & \\
 & M &
\end{array}
\qquad (2.1)
$$

where $\pi{:}M \times U \to M$ is the canonical projection. In local coordinates (x, u) for $M \times U$, one has, with a slight abuse of notation, $u = \gamma(x, v)$ where v is the new input. Since γ is nonsingular, feedback can (and will) be viewed simply as a state-dependent change of the input coordinates.

Invariant and controlled invariant equivalence relations are now introduced.

Definition 2.3: Let $\Sigma(M \times U, M, f)$ be a nonlinear discrete-time system and let R be an equivalence relation [31] on M. R is said to be *invariant* if for each $u \in U$, $f(x, u)Rf(\bar{x}, u)$ whenever $xR\bar{x}$. R is said to be *controlled invariant* if there exists a feedback function γ such that R is invariant for the feedback modified (or closed-loop) system $\Sigma(M \times U, M, f{\circ}\gamma)$.

Example 2.2: Let $\Sigma(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n, Ax + Bu)$ denote a discrete-time linear system on $\mathbb{R}^n$ and let V be a subspace of $\mathbb{R}^n$. Then V induces an equivalence relation on $\mathbb{R}^n$ by defining $xR\bar{x}$ if $x - \bar{x} \in V$. The condition $f(x, u)Rf(\bar{x}, u)$ amounts to $A(x - \bar{x}) \in V$ whenever $x - \bar{x} \in V$. This is equivalent to $AV \subset V$. In other words, V is an invariant subspace of A.

The above example demonstrates that Definition 2.3 is in fact an abstraction of the usual notion of (A, B) invariance for discrete-time linear systems. Hence, it should be clear that such invariant equivalence relations ought to play an important role in understanding the structure of discrete-time nonlinear systems and in solving certain synthesis problems; in this regard, one ought to glance at Proposition 4.1. However, since they involve global computations, except in certain special cases, one has little hope of actually performing the necessary calculations. This motivates trying to localize the above notion in a useful way. For those equivalence relations arising from a regular foliation $\mathfrak{F}$ on M, it turns out that this is possible. (The reader may wish at this time to consult [3, pp. 332–333] for a discussion on distributions and foliations.)

Towards this end, consider a distribution Δ on M which gives rise to a *regular foliation* $\mathfrak{F}$ on M in the sense that its maximal integral manifolds partition M into a disjoint union of fixed-dimensional submanifolds. Then $\mathfrak{F}$ induces an equivalence relation on M; namely, $xR\bar{x}$ if x and $\bar{x}$ are elements of the same leaf of $\mathfrak{F}$. $\mathfrak{F}$ will be said to be *locally invariant* if for each $u \in U$ and x_o

$\in M$ there exists an open set θ about x_o such that $f(x, u)Rf(\bar{x}, u)$ whenever $x, \bar{x} \in \theta$ and $xR\bar{x}$. For a regular foliation $\mathfrak{F}$, this can be characterized as follows. For a vector field $X \in \Delta$, let $X_t(\cdot)$ denote its flow. Then $\mathfrak{F}$ is locally invariant if and only if for each such X there exists an $\epsilon > 0$ such that $f(X_t(x), u)Rf(x, u)$ for all $|t| < \epsilon$. Differentiating with respect to t and evaluating at $t = 0$ then gives

$$f_*(\cdot, u)X \subset \Delta. \qquad (2.2)$$

This leads to the following definition.

Definition 2.4: Let Δ be an involutive distribution on M (possibly not giving rise to a regular foliation). Then Δ is an *invariant distribution* of $\Sigma(M \times U, M, f)$, if

$$f_*(\cdot, u)\Delta \subset \Delta. \qquad (2.3)$$

Δ is *(locally) controlled invariant* if there exists (locally) a feedback γ such that Δ is invariant for $f{\circ}\gamma$.

Example 2.3:

a) Let Σ and V be as in Example 2.2. V can be regarded as a constant distribution on $\mathbb{R}^n$ and then (2.3) becomes $AV \subset V$.

b) Let $(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n, f)$ be a discrete-time system on $\mathbb{R}^n$ and let Δ be a distribution on $\mathbb{R}^n$. Then in the usual rectangular coordinates for $\mathbb{R}^n$, (2.3) becomes

$$\frac{\partial f(x, u)}{\partial x} \nu \subset \Delta(f(x, u)) \qquad \text{for each } \nu \in \Delta(x). \quad (2.4)$$

Remark 2.1: Invariant foliations for autonomous discrete-time systems (i.e., no inputs) were introduced in [32]. There it was also noted that invariant foliations, in a neighborhood of a fixed point of the dynamics, induce decompositions of the systems' dynamics and that such decompositions have been important in the solution of various synthesis problems for continuous-time systems.

Doing local coordinate calculations for discrete-time systems can be more delicate than in the case of continuous-time systems since $f(\cdot, u)$ applied to a point x in a local coordinate chart may leave the chart even for "small" inputs.[2] Thus, one must introduce a pair of coordinate charts: one in the domain of f and another in its image. This paragraph will develop a convenient pair of charts for doing calculations and will introduce some useful notation. Let $(x_o, u_o) \in M \times U$, and consider $f(x_o, u_o) \in M$. Choose a coordinate chart $(\tilde{V}_M, \tilde{\phi}_M)$ and $f(x_o, u_o)$ and consider the open set $f^{-1}(\tilde{V}_M)$ about (x_o, u_o). Now choose a coordinate chart $(V_{M \times U}, \phi_{M \times U})$ about (x_o, u_o) such that $V_{M \times U} \subset f^{-1}(\tilde{V}_M)$. $(V_{M \times U}, \phi_{M \times U})$, $(\tilde{V}_M, \tilde{\phi}_M)$ will be called a *coordinate chart pair*. Denote coordinates for $(V_{M \times U}, \phi_{M \times U})$ by (x, u), and for $(\tilde{V}_M, \tilde{\phi}_M)$ by x. This abuse of notation is useful and permits one to perform local calculations as if one were working in a single coordinate chart. The coordinate chart pair will simply be denoted by (x, u). If Δ is an involutive distribution on M having constant dimension, then by the Frobenius theorem one can assume that $\Delta = \text{span} \{\partial/\partial x^1, \cdots, \partial/\partial x^k\}$ in each chart $(V_{M \times U}, \phi_{M \times U})$, $(\tilde{V}_M, \tilde{\phi}_M)$. Hence, the notation $\Delta = \text{span} \{\partial/\partial x^1, \cdots, \partial/\partial x^k\}$ is not ambiguous. If it should become important to distinguish between the different domains of x, this could be done via the open sets $V_{M \times U}$ and $\tilde{V}_M$. Note that $\pi(V_{M \times U})$ and $\tilde{V}_M$ may or may not intersect, and may or may not coincide. (Recall $\pi{:}M \times U \to M$ is the canonical projection.) In a neighborhood of a fixed point, $f(x_o, u_o) = x_o$, one may always choose $V_{M \times U}$ such that $\pi(V_{M \times U}) \subset V_M$ and $\pi{\circ}\phi_{M \times U} = \tilde{\phi}_M|\pi(V_{M \times U})$, so that one is essentially using a single coordinate chart.

As a consequence of the Frobenius theorem and the definition of an invariant distribution, one has the following result which shows that invariant distributions are intimately linked to the classical notion of a subsystem.

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[2] If one is considering systems defined on $\mathbb{R}^n$ which are represented in globally defined coordinates (such as rectangular coordinates), then the following considerations are unnecessary.

Proposition 2.1: Let $\Sigma(M \times U, M, f)$ be a nonlinear discrete-time control system and let Δ be an involutive constant-dimensional invariant distribution. Then for any $x_o \in M$ and $u_o \in U$ there exists a coordinate chart pair (x, u) in which Σ has the form

$$x_{k+1}^1 = f_1(x_k^1, u_k)$$
$$x_{k+1}^2 = f_2(x_k^1, x_k^2, u_k) \qquad (2.5)$$

where (x^1, x^2) is a suitable partition of x.

Conditions will now be established under which an involutive distribution Δ is locally controlled invariant. In the following, $V(M \times U)$ denotes the vertical distribution on $M \times U$, i.e., $V(M \times U) = \{X \in T(M \times U) | \pi_*(X) = 0\}$.

Lemma 2.1: Let $\Sigma(M \times U, M, f)$ be a nonlinear discrete-time control system and let Δ be a regular (i.e., involutive and constant dimensional) distribution on M. Then the following are equivalent.

a) Locally there exists a regular distribution E on $M \times U$ such that $\pi_* E = \Delta$, dim E = dim Δ, and $f_* E \subset \Delta$.

b) There exists a coordinate chart pair in which Δ is invariant.

c) Δ is locally controlled invariant.

d) Locally there exists a regular distribution F on $M \times U$ such that $\pi_* F = \Delta$ and $f_* F \subset \Delta$.

Proof:

a) $\Rightarrow$ b): Let (x, u) be a coordinate chart pair such that $\Delta = $ span $\{\partial/\partial x^1, \cdots, \partial/\partial x^k\}$. Since $\pi_*: E \to \Delta$ is an isomorphism, there exists a (possibly state-dependent) change of coordinates on U, resulting in coordinates (x, v), such that $E = $ span $\{\partial/\partial x^1, \cdots, \partial/\partial x^k\}$. Hence, $f_* E = f_* \Delta \subset \Delta$.

b) $\Rightarrow$ c): Let (x, u) be a coordinate chart pair. By b), there exists another coordinate chart pair (x, v) such that Δ in these coordinates is invariant. Let γ be the change of coordinates between (x, u) and (x, v). Then γ is the required feedback.

c) $\Rightarrow$ d): Simply let $F = \gamma_* \Delta$.

d) $\Rightarrow$ a): Let (x, u) be a coordinate chart pair such that $\Delta = $ span $\{\partial/\partial x^1, \cdots, \partial/\partial x^k\}$. Consider $F \cap V(M \times U)$. Then dim $F = $ dim $\Delta + $ dim $(F \cap V(M \times U))$, and since dim F and dim Δ are constant, it follows that dim $(F \cap V(M \times U))$ is also. Furthermore, it is involutive since it is the intersection of two involutive distributions. Thus, $F \cap V(M \times U)$ is a regular distribution and one can integrate it to obtain new input coordinates $(v^1, \cdots, v^m)$ such that $F \cap V(M \times U) = $ span $\{\partial/\partial v^{r+1}, \cdots, \partial/\partial v^m\}$. Since $\pi_* F = \Delta$, in these coordinates, $F = $ span $\{\partial/\partial x^j + \Sigma_{i=1}^r \alpha_i^j (x, v) \partial/\partial v^i, \partial/\partial v^{r+1}, \cdots, \partial/\partial v^m | j = 1, \cdots, k\}$ for some functions α_i^j. Let $E = $ span $\{\partial/\partial x^j + \Sigma_{i=1}^r \alpha_i^j (x, v) \partial/\partial v^i | j = 1, \cdots, k\}$. From the involutivity of F, one deduces that of E. Clearly the dimension of E equals that of Δ. Since $\pi_* (F \cap V(M \times U)) = 0$, $\pi_* E = \Delta$. Since $E \subset F, f_* E \subset \Delta$. In passing, note that the involutivity of F implies that $\alpha_i^j (x, v)$ does not depend on $(v^{r+1}, \cdots, v^n)$. $\square$

The following theorem gives a simple coordinate-independent test for checking the local controlled invariance of a given distribution. Its proof is modeled on a theorem in [18] which gives the corresponding result in continuous time.

Theorem 2.1: Let $\Sigma(M \times U, M, f)$ be a nonlinear discrete-time system and let Δ be an analytic involutive constant-dimensional distribution. Then if Δ is a locally controlled invariant

$$f_*(\pi_*^{-1}(\Delta)) \subset \Delta + f_*(V(M \times U)). \qquad (2.6)$$

Conversely, if (2.6) holds and if both $f_*^{-1}(\Delta) \cap V(M \times U)$ and $f(x, \cdot): U \to M$ have constant rank, then Δ is locally controlled invariant.

Remark 2.2: The constant rank assumptions *will* hold on an open dense subset of $M \times U$ and on such a subset (2.6) is then both necessary and sufficient.

Proof of Theorem 2.1: Assume Δ is locally controlled invariant. Then by Lemma 2.1 there exists a coordinate chart pair (x, u) in which Δ is invariant. In these coordinates one easily verifies that (2.6) holds. But (2.6) is coordinate independent, and hence the necessity is shown.

For the sufficiency, let (x, u) be a coordinate chart pair such that $\Delta = $ span $\{\partial/\partial x^1, \cdots, \partial/\partial x^k\}$. Denote the components of f by $(f^1, \cdots, f^n)$. Then under the constant rank hypotheses (2.6) is equivalent to the local existence of analytic functions $\alpha_i^j(x, u)$ such that

$$\frac{\partial f^l}{\partial x^j}(x, u) + \sum_{i=1}^m \frac{\partial f^l}{\partial u^i}(x, u)\alpha_i^j(x, u) = 0 \qquad (2.7)$$

for $l = k + 1, \cdots, n$ and $j = 1, \cdots, k$. Let P be the annihilating codistribution of Δ (i.e., ker $P = \Delta$). Then both $f^* P$ and $\pi^* P$ are involutive codistributions on $M \times U$, and moreover, $Q := f^* P + \pi^* P$ is also involutive (since the intersection of two involutive distributions is involutive). In the above local coordinates one has $P = $ span $\{dx^{k+1}, \cdots, dx^n\}$, and $Q = $ span $\{dx^{k+1}, \cdots, dx^n, \Sigma_{i=1}^k (\partial f^{k+1}/\partial x^i) dx^i + \Sigma_{j=1}^m (\partial f^{k+1}/\partial u^j) du^j, \cdots, \Sigma_{i=1}^k (\partial f^n/\partial x^i) dx^i + \Sigma_{j=1}^m (\partial f^n/\partial u^j) du^j\}$. From (2.7), it follows that ker $Q = $ span $\{\partial/\partial x^j + \Sigma_{i=1}^m \alpha_i^j \partial/\partial u^i; j = 1, \cdots, k\} + $ ker span $\{\Sigma_{j=1}^m (\partial f^{k+1}/\partial u^j) du^j, \cdots, \Sigma_{j=1}^m (\partial f^n/\partial u^j) du^j, dx^1, \cdots, dx^n\}$. This last term is precisely $V(M \times U) \cap f_*^{-1}(\Delta)$, and hence ker Q has constant dimension. Thus, letting $E = $ ker Q, one has that E is a regular distribution on $M \times U$ such that $\pi_* E = \Delta$ and $f_*(E) \subset \Delta$. Therefore, by d) of Lemma 2.1, Δ is locally controlled invariant. $\square$

Example 2.4:

a) Let $\Sigma(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n, f)$ be a nonlinear discrete-time system on $\mathbb{R}^n$, and let Δ be an analytic involutive constant-dimensional distribution on $\mathbb{R}^n$. Then in standard rectangular coordinates (2.6) can be rewritten as

$$\frac{\partial f(x, u)}{\partial x} \nu \in \Delta(f(x, u)) + \text{span} \left\{ \frac{\partial f(x, u)}{\partial u} \right\} \qquad (2.8)$$

for each $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ and $\nu \in \Delta(x)$. The constant rank hypotheses are that span $\{\alpha \in \mathbb{R}^m | \partial f(x, u)/\partial u \, \alpha \in \Delta(f(x, u))\}$ and span $\{\partial f(x, u)/\partial u\}$ each have constant dimension. Due to analyticity, this will always be true at least "almost everywhere."

b) For a linear system $\Sigma(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n, Ax + Bu)$ and subspace $V \subset \mathbb{R}^n$, (2.6) becomes

$$AV \subset V + \text{Im } B. \qquad (2.9)$$

The constant rank hypotheses are that Im$B \cap V$ and ImB have constant rank, which are of course always true.

III. MAXIMAL CONTROLLED INVARIANT DISTRIBUTIONS

In this section, a slightly weaker notion than local controlled invariance will be introduced. It will be shown that with respect to this weaker property, a maximal such distribution contained in a given distribution exists. This will be the nonlinear analog of the linear result that a maximal (A, B) invariant subspace contained in a given subspace exists. An algorithm for calculating these maximal distributions will be given.

Definition 3.1: A distribution Δ is said to satisfy the LCI (local controlled invariance) condition if

$$f_*(\pi_*^{-1}(\Delta)) \subset \Delta + f_*(V(B)) \qquad (3.1)$$

on an open dense subset of $M \times U$.

The following lemma is important.

Lemma 3.1:

a) Let Δ_1 and Δ_2 both satisfy the LCI condition; then so does their sum $\Delta_1 + \Delta_2$.

b) Let Δ satisfy the LCI condition; then so does its involutive closure $\bar{\Delta}$.

Proof: a) is true by the linearity of f_*. For b), let X_1, $X_2 \in \pi_*^{-1}(\Delta)$, and note that X_i can be written in the form $X_i = \alpha_i(x) \, \partial/\partial x + \beta_i(x, u) \, \partial/\partial u$ in a local coordinate chart. Equation (3.1) implies that on an open dense subset of $M \times U$ there exist vector fields $Y_i \in V(M \times U)$ such that $f_*(X_i - Y_i) \subset \Delta$ for $i = 1, 2$. Thus, $[X_1 - Y_1, X_2 - Y_2] \subset f_*^{-1}(\Delta)$ which equals $f_*^{-1}(\bar{\Delta})$ on an open dense subset. But since $[X_i, Y_j] \in V(M \times U)$, due to the form of X_i, and since $[Y_1, Y_2] \in V(M \times U)$, it follows that $f_*[X_1, X_2] \in \bar{\Delta} + f_*(V(M \times U))$ on an open dense subset of $M \times U$. $\square$

This yields the following result.

Theorem 3.1: Let K be an involutive distribution on M. Then K contains a maximal distribution Δ^* satisfying the LCI condition; moreover, Δ^* is necessarily involutive and on the open dense subsets of $M \times U$ and M where the constant rank hypotheses of Theorem 2.1 are satisfied, it is locally controlled invariant.

Proof: Define a linear partial ordering on those distributions that satisfy the LCI condition by $\Delta_1 < \Delta_2$ if $\Delta_1 \subset \Delta_2$ (pointwise). Then Lemma 3.1 and Zorn's lemma [33] yield the first part of the result, once one notes that the zero distribution satisfies the LCI condition. The second part follows from Theorem 2.1. $\square$

It can now be seen why the LCI condition was introduced: if Δ_1 and Δ_2 were regular locally controlled invariant distributions, $\bar{\Delta}_1 + \bar{\Delta}_2$ may not be, since its dimension may not be constant, and hence Zorn's lemma could not be applied. However, using Lemma 2.1 the following proposition can be shown.

Proposition 3.1:

a) Let Δ_1 and Δ_2 be two regular locally controlled invariant distributions such that dim $\overline{\Delta_1 + \Delta_2}$ is constant. Then $\overline{\Delta_1 + \Delta_2}$ is locally controlled invariant.

b) Let Δ_1 and Δ_2 be two analytic locally controlled invariant distributions. Then $\overline{\Delta_1 + \Delta_2}$ is locally controlled invariant on an open dense subset.

An algorithm will now be given for calculating the maximal distribution satisfying the LCI condition which is contained in a given involutive distribution. Although such distributions are not the objects for which one searches when trying to solve synthesis problems, they are nevertheless important since: 1) one will have obtained a maximal locally controlled invariant distribution on an open dense subset; 2) any regular locally controlled invariant distribution is contained in one; 3) under certain constant rank conditions it *is* the maximal locally controlled invariant distribution.

The D*-Algorithm

Let K be an involutive analytic distribution on M. Define the following sequence of distributions on $M \times U$:

$$D^o = \pi_*^{-1}(K)$$

$$D^{m+1} = \{X \in D^m | f_*X \subset \pi_*D^m + f_*(V(M \times U)) \text{ on an}$$

$$\text{open dense subset of } M \times U\}$$

and set $D^* = \lim_{m \to \infty} D^m$, $\Delta^* = \pi_*D^*$. $\square$

Theorem 3.2: The D^*-algorithm possesses the following properties.

a) It is well defined, each D^m is involutive, and the indicated limits D^* and Δ^* exist.

b) Δ^* is the maximal distribution contained in K which satisfies the LCI condition.

c) If at each step D^m has constant dimension, then $D^* = D^k$, where $k = \dim K$.

d) In any case, $D^* = D^k$ on an open dense subset, where $k = \dim K$.

Proof:

a) D^o is clearly a well-defined involutive distribution which projects well to M. Assume that D^{m-1} is such a

distribution. Then the linearity of f_* gives that D^m is a well-defined distribution. The proof that D^m is involutive is the same as part b) of Lemma 3.1. It is clear that $V(M \times U) \subset D^m$. Thus, D^m is an involutive distribution on $M \times U$ containing the vertical vector fields, and hence π_*D^m is well defined. D^* exists since the D^m's are nested. Δ^* exists since D^* is clearly involutive and contains the vertical vector fields.

b), c), and d) follow the same lines as [34, Theorem 4.1]. $\square$

Example 3.1:

a) Let $\Sigma(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n, f)$ be a nonlinear discrete-time system on $\mathbb{R}^n$. A distribution Δ satisfies the LCI condition if (2.8) holds "almost everywhere" (the reason for the almost everywhere is purely technical). The D^*-algorithm can be rewritten as

$$\Delta^0 = K$$

$$\Delta^{m+1} = \{X \in \Delta^m \mid \frac{\partial f(x, u)}{\partial x} X \subset \Delta^m(f(x, u)) + \text{Im} \frac{\partial f(x, u)}{\partial u}$$

$$\text{for almost all } (x, u) \in \mathbb{R}^n \times \mathbb{R}^m\}$$

$$\Delta^* = \lim_{m \to \infty} \Delta^m.$$

b) Let $\Sigma(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n, Ax + Bu)$ be a linear system and $V \subset \mathbb{R}^n$ be a subspace. Then V satisfies the LCI condition if and only if it is (A, B) invariant. The D^*-algorithm is then the usual V^*-algorithm [1, p. 91].

IV. DISTURBANCE DECOUPLING

This section will formulate a disturbance decoupling problem for the class of discrete-time systems considered in this paper. The property of being disturbance decoupled will be shown to be equivalent to the existence of a certain invariant equivalence relation. This will lead to a natural local version of the disturbance decoupling problem for discrete-time systems which involves invariant distributions. The results of Sections II and III will be used to give necessary and sufficient conditions for its solvability. This last part will parallel the result of [9] for continuous-time systems.

Definition 4.1: A discrete-time *nonlinear control system with disturbances* is a 4-tuple $\Sigma(W, M \times U, M, f)$, where W, M, and U are analytic manifolds and $f: M \times U \times W \to M$ is an analytic mapping. As before, M and U are the state and control spaces, respectively, W is the space of disturbances. The dynamics are given by $x_{k+1} = f(x_k, u_k, w_k)$. If *outputs* are present, that is, if there is an analytic mapping $h: M \to Y$, Y being an analytic manifold, then the notation $\Sigma(W, M \times U, M, f, Y, h)$ will be used. The output is then $y_k = h(x_k)$. In the following, $\tilde{\pi}$ will denote the projection from $M \times U \times W \to M \times U$ and π' will denote the mapping $\pi \circ \tilde{\pi}$ so that $\pi': M \times U \times W \to M$.

In this context state variable feedback becomes the following.

Definition 4.2: A *feedback* function for a system with disturbances is a diffeomorphism of the form $(\gamma, Id \circ \hat{\pi})$, where $\gamma: M \times U \to M \times U$ is a feedback in the sense of Definition 2.2, $Id: W \to W$ is the identity function, and $\hat{\pi}: M \times U \times W \to W$ is the canonical projection. In local coordinates (x, u, w) one has, with a slight abuse of notation, $f \circ \gamma(x, u, w) = f(x, \gamma(x, u), w)$. Note that this definition reflects the (implicit) assumptions that one can neither measure nor modify the disturbances directly.

A system $\Sigma(W, M \times U, M, f, Y, h)$ is said to be *disturbance decoupled* if the output does not depend upon the disturbances. More precisely, this means that for each initial condition x_o, time k, input sequence $(u_i)_{i=1}^\infty$, and for arbitrary disturbance sequences $(w_i)_{i=1}^\infty$ and $(\tilde{w}_i)_{i=1}^\infty$, one has

$$y_k(x_o, u_1, \cdots, u_{k-1}, w_1, \cdots, w_{k-1})$$

$$= y_k(x_o, u_1, \cdots, u_{k-1}, \tilde{w}_1, \cdots, \tilde{w}_{k-1}) \quad (4.1)$$

where $y_k = h(x_k)$, for the x_k's defined in the obvious way. The *disturbance decoupling problem* is to find, if it exists, a feedback function γ such that $\Sigma(W, M \times U, M, f \circ \gamma)$ is disturbance decoupled.

Proposition 4.1: $\Sigma(W, M \times U, M, f, Y, h)$ is disturbance decoupled if and only if there exists an equivalence relation R on M such that, whenever $xR\bar{x}$, one has that

$$h(x) = h(\bar{x}) \qquad (4.2)$$

and

$$f(x, u, w) R f(\bar{x}, u, \bar{w})$$

for all u, and for arbitrary w and $\bar{w}$.

Proof: Assume there exists an equivalence relation R on M satisfying (4.2). Let M/R denote the set of equivalence classes. Then (4.2) gives that Σ projects, in a set theoretic sense, to a system on M/R

$$\bar{x}_{k+1} = \bar{f}(\bar{x}_k, u)$$

$$y_k = \bar{h}(\bar{x}_k)$$

having the same input–output mapping as Σ, from which it is clear that y_k does not depend upon the disturbances. On the other hand, for fixed x and $\bar{x}$, define $xR\bar{x}$ if, for arbitrary sequences $(u_i)_{i=1}^{\infty}$, $(w_i)_{i=0}^{\infty}, (\bar{w}_i)_{i=0}^{\infty}$,

$$y_k(x, u_0, \cdots, u_{k-1}, w_0, \cdots, w_{k-1})$$

$$= y_k(\bar{x}, u_0, \cdots, u_{k-1}, \bar{w}_0, \cdots, \bar{w}_{k-1})$$

for all $k = 0, 1, 2, \cdots$ for the y_k's defined in the obvious way. Then it is easily checked that R is an equivalence relation satisfying (4.2). $\square$

Remark 4.1: For linear systems, one can show that R corresponds to a linear subspace of the state space. For nonlinear systems an important question, not addressed here, is when are the orbits of R (locally immersed) submanifolds of M? In other words, when does R correspond to a foliation induced from a distribution? (It appears that Sard's theorem may be useful here.)

Hence, to solve the disturbance decoupling problem, one must give conditions for the existence of a feedback function γ so that, the closed-loop system will admit an equivalence relation satisfying (4.2). However, since such conditions would necessarily involve global computations, one is led to localizing the problem. Towards this end, assume that the equivalence relation R of Proposition 4.1 comes from a foliation $\mathfrak{F}$ with associated distribution Δ. Then an easy calculation gives that (4.2) is locally equivalent to: 1) $h_*(\Delta) = 0$, 2) $f(\cdot, u, w)_*\Delta \subset \Delta$, and 3) $f_*(TW) \subset \Delta$. This leads to the following definition.

Definition 4.3: The disturbance decoupling problem is *locally regularly solvable* for a system $\Sigma(W, M \times U, M, f, Y, h)$ if there locally exists a feedback γ and a regular distribution Δ such that

a) $h_*(\Delta) = 0$;

b) $(f \circ \gamma)(\cdot, u, w)_*\Delta \subset \Delta$;

c) $(f \circ \gamma)_*(TW) \subset \Delta$.

Remark 4.2:

a) Let (x, u, w) be a local coordinate chart pair. Then the above gives that $\tilde{f} = f \circ \gamma$ has the following local form:

$$x_{k+1}^1 = \tilde{f}_1(x_k^1, u_k)$$

$$x_{k+1}^2 = \tilde{f}_2(x_k^1, x_k^2, u_k, w_k)$$

$$y_k = h(x_k^1) \qquad (4.3)$$

for properly chosen (x^1, x^2).

b) If by a *regular* solution to the disturbance decoupling problem [3] one means that there exists a regular foliation $\mathfrak{F}$ such that $M/\mathfrak{F}$ is a smooth manifold and the system $f \circ \gamma$ projects to a smooth system $\tilde{f}$ on $M/\mathfrak{F}$, then Definition 4.3 clearly expresses the local necessary conditions. On the other hand, (4.3) shows that the system is locally disturbance decoupled.

The results from Sections II and III will now be used to give conditions for the local regular solvability of the disturbance decoupling problem for discrete-time systems.

Theorem 4.1: Let $\Sigma(W, M \times U, M, f, Y, h)$ be a nonlinear discrete-time system with disturbances. Let $K = \ker h_*$ and let Δ^* be the maximal LCI distribution contained in K. Then on the open dense subsets where $\Delta^*, f_*^{-1}(\Delta^*) \cap (V(M \times U) \times TW)$ and $f(x, \cdot, \cdot): U \times W \to M$ have constant rank, the disturbance decoupling problem is locally regularly solvable if and only if $f_*(TW) \subset \Delta^*$.

Proof: The necessity is clear once one notes that c) of Definition 4.3 is equivalent to $f_*(TW) \subset \Delta^*$ since the feedback function cannot depend upon the disturbances.

For the sufficiency, let $\tilde{B} = M \times U \times W$ and recall that $\pi': \tilde{B} \to M$ is the natural projection (see Definition 4.1). Let E be the regular distribution on $\tilde{B}$ satisfying $\pi'_* E = \Delta^*$ and $f_*(E) \subset \Delta^*$, which the proof of Theorem 2.1 constructs for the system $\Sigma(\tilde{B}, M, f)$. Since $f_*(TW) \subset \Delta^*$, it follows that $TW \subset E$, and hence $E' = \tilde{\pi}_* E$ is a regular distribution on $M \times U$. Moreover, $\pi_* E' = \pi_* \circ \tilde{\pi}_* E = \pi'_* E = \Delta^*$, and $f_*(E') \subset f_*(E' \times TW) = f_*(E) \subset \Delta^*$. Thus, E' is a regular distribution on $M \times U$ satisfying all of the conditions of part c) of Lemma 2.1, and therefore a feedback γ can be constructed independently of the disturbance which makes Δ^* invariant. (Note that Δ^* was automatically controlled invariant, but that the required feedback could have depended on the disturbances.) Thus, b) of Definition 4.3 is fulfilled; a) is satisfied since by construction $\Delta^* \subset \ker h_*$. Finally, c) follows from $f_*(TW) \subset \Delta^*$ since γ does not depend on the disturbances. $\square$

Example 4.1:

a) Consider a nonlinear discrete-time system with disturbances $x_{k+1} = f(x_k, u_k, w_k)$, $y_k = h(x_k)$, defined on $\mathbb{R}^n$. To locally solve the disturbance decoupling problem, the key object is the maximal locally controlled invariant distribution contained in the kernel of the output function h. Let $K(x) = \ker h_*(x) = \{v \in \mathbb{R}^n | (\partial h(x)/\partial x) v = 0\}$, and let Δ^* be the maximal locally controlled invariant distribution contained in K. Then under certain constant rank hypotheses (see Theorem 4.1 and Example 2.4), the disturbance decoupling problem is locally solvable if and only if $\partial f(x, u, w)/\partial w \subset \Delta^*(f(x, u, w))$.

b) The above result should be compared to the linear one; namely, let $x_{k+1} = Ax_k + Bu_k + Dw_k$, $y_k = Cx_k$ be a linear system with disturbances, then the disturbance decoupling problem is solvable if and only if $\operatorname{Im}D \subset V^*$, where V^* is the maximal (A, B) invariant subspace contained in the kernel of C.

Remark 4.3: In [35]–[37], discrete-time nonlinear control systems are studied from an input/output point of view via their generating series. It has been announced in [35] that the disturbance decoupling problem can be solved using such methods, although the details will appear elsewhere.

V. Conclusions and Comments

The basis for a geometric approach to solving certain synthesis problems for discrete-time nonlinear control systems has been developed. The key element was the introduction of invariant distributions. Necessary and sufficient conditions were given for a distribution to be invariant after feedback, i.e., to be locally controlled invariant. In addition, certain maximality properties were established for locally controlled invariant distributions, and an algorithm was given for their calculation. Finally, these results were applied to the disturbance decoupling problem to give necessary and sufficient conditions for its local solvability.

The reader has probably noted that, with the exception of linear

systems, "affine" discrete-time systems of the form $x_{k+1} = f_o(x_k) + \Sigma_{i=1}^{m} u_i f_i(x_k)$ were never considered explicitly. This is because if a system Σ is affine in coordinates (x, u), a nonlinear change of coordinates $(\phi(x), u)$ will result in a "nonaffine" system. Hence, from a geometric point of view, discrete-time *affine systems* are not "natural"; this is in contrast to the situation of continuous-time systems where "affineness" is well defined. On the other hand, if U is a vector space, the notation of *affine feedback* is well defined. Hence, under what conditions can the feedback required to achieve controlled invariance be taken to be affine (i.e., $\gamma(x, u) = \alpha(x) + \beta(x)u$), is a well-posed and interesting question. The answer appears to be intimately related to the distribution E on $M \times U$ associated to a distribution Δ on M, and will be addressed elsewhere.

Finally, the results of this paper can be easily extended to a more general class of systems by everywhere replacing $\pi:M \times U \to M$ with a general fiber bundle $\pi:B \to M$. Assume that B is equipped with an integrable connection H [38]. Then a distribution Δ is said to be invariant if $f_* X^l \subset \Delta$ for all $X \in \Delta$ where X^l is the horizontal lift of X. Lemma 2.1 and Theorem 2.2 can then be interpreted as giving the necessary and sufficient conditions for the local existence of a connection on B with respect to which Δ is an invariant distribution. The relations between feedback and connections are nicely treated in [17].

APPENDIX

This Appendix considers the problem of disturbance decoupling with partially measurable disturbances; this is motivated by [39] which addresses an analogous problem in continuous time which arose in a practical application. One seeks a feedback which depends on the state, controls, and measured disturbances and which renders the closed-loop system disturbance decoupled. So assume the disturbance space can be factored as $W = W^1 \times W^2$, where W^1 represents the measured disturbances.

Theorem A: Let $\Sigma(W^1 \times W^2, M \times U, M, f, Y, h)$ be a discrete-time nonlinear system with partially measurable disturbances. Let $K = \ker h_*$ and let Δ^* be the maximal LCI distribution contained in K. Then on the open dense subsets of M and $M \times U$ where Δ^*, $f_*^{-1}(\Delta^*) \cap (V(M \times U) \times TW^1 \times TW^2)$ and $f(x, \cdot, \cdot):U \times W \to M$ have constant rank, the disturbance decoupling problem with partial measurements of the disturbances is locally regularly solvable if and only if $f_*(TW^1) \subset \Delta^* + f_*(V(M \times U))$ and $f_*(TW^2) \subset \Delta^*$.

Proof: The necessity is easy. For the sufficiency, let E be the regular distribution on $\tilde{B} = M \times U \times W^1 \times W^2$ satisfying $\pi'_* E = \Delta^*$ and $f_* E \subset \Delta^*$, which the proof of Theorem 2.1 constructs for the system $\Sigma(\tilde{B}, M, f)$. To guarantee that γ can be constructed independent of W^2, it must be shown that E projects well to $M \times U \times W^1$; but this follows from $f_*(TW^2) \subset \Delta^*$ for it implies that $TW^2 \subset E$. To further ensure that γ can be constructed such that the "W^1-component" is the identity (i.e., the feedback $\gamma:M \times U \times W^1 \to M \times U \times W^1$ does not modify the disturbances), one must show that E projects well to W^1; but since $f_*(TW^1) \subset \Delta^* + f_*(V(M \times U))$, one obtains the existence of a vector of functions such that $\partial/\partial w^1 + m(x, u, w) \partial/\partial u \in E$, and hence $\pi_{1*} E = TW^1$, where $\pi_1:M \times U \times W^1 \times W^2 \to W^1$ is the canonical projection. $\square$

Remark A: Let $x_{k+1} = Ax_k + Bu_k + D^1 w_k^1 + D^2 w_k^2$, $y = Cx_k$ be a linear system with partial measurements of the disturbances. Let V^* be the maximal (A, B) invariant subspace contained in the kernel of C. Then the disturbance decoupling problem with partial measurements of the disturbances is solvable if and only if span $D^1 \subset V^* + $ span B and span $D^2 \subset V^*$.

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Jessy W. Grizzle (S'78-M'83), for a photograph and biography, see p. 258 of the March 1985 issue of this TRANSACTIONS.