*Automatica*, Vol. 26, No. 6, pp. 997–1007, 1990
Printed in Great Britain.

0005-1098/90 $3.00 + 0.00
Pergamon Press plc


# Sampled-data Observer Error Linearization*

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*An analysis of the effects of time-sampling on the observer error linearization design methodology shows that requiring the method to be applicable for an open set of sampling times trivializes the class of allowable systems.*

**Key Words**—Control theory; digital control; nonlinear systems; observers; sampled data systems.

**Abstract**—The effects of time-sampling on the solvability conditions for the observer linearization design methodology are investigated. It is shown that the class of systems for which this design method can be applied for an open set of sampling times is quite small. In particular, when the dimension of the state space is two, it consists only of those systems that are state-equivalent to a linear system. The practical implication is that digital implementations of this methodology will have to be approximate.

## 1. INTRODUCTION

RECENTLY, there has been considerable interest in the design of nonlinear observers. One approach has focused on identifying a class of nonlinear systems that can be transformed into linear systems through the application of output injection and state coordinate transformations. In this case, the error dynamics of the observer is (exactly) linearizable and the observer design theory for the class of linear systems can be applied.

This design methodology, called observer linearization, was proposed independently by Krener and Isidori (1983) and Bestle and Zeitz (1983) for the class of scalar output systems. The extension to systems with multiple outputs, and to systems with inputs, has been done by Krener and Respondek (1985). Xia and Gao (1988, 1989) correct an error in Krener and Respondek (1985) and also give a new necessary and sufficient condition for the solvability of the observer linearization problem for time-varying systems.

Here, we wish to investigate the effect of time-sampling on the observer design methodology of Krener and Isidori. In particular, we want to understand when the methodology is *robust* with respect to the introduction of sampling, that is, when the procedure can be successfully applied to sampled-data representations of a plant for an *open set of sampling times.* This property is crucial from an engineering design point of view because otherwise, the probability of choosing a "good" sampling time would be nil.

Our results can be viewed as being dual to those of Arapostathis *et al.* (1989) on the feedback linearization problem, which were motivated by an example in Grizzle (1986).

In Section 2, we summarize the main results of Krener and Isidori (1983). In Section 3, we show that their results carry over to the class of discrete-time systems with only minor modifications; one hindrance, however, is that there is no equivalent of the Jacobi identity. Section 4 contains the main results. It is first shown that there exists an open set of sampling times for which sampled-data representations of a system are locally state-equivalent to a linear system if, and only if, the underlying continuous time system is locally state-equivalent to a linear system. Next we establish that a certain set of parameterized linear equations with a singularity admits an analytic solution. This technical result is crucial for proving that whenever there exists an open set of sampling times for which the observer linearization problem is solvable for sampled-data representations of a given plant, the problem must also be solvable for a continuous-time representation of the plant. This gives a *necessary* condition for the "robust" solvability of the observer linearization problem with respect to time-sampling. We then show that this condition is far from sufficient because we prove that, when the dimension of the state

* Received 31 March 1989; revised 7 December 1989; received in final form 2 January 1990. The original version of this paper was not presented at any IFAC meeting. Supported by the National Science Foundation under Grant ECS-88-96136. This paper was recommended for publication in revised form by Associate Editor V. I. Utkin under the direction of Editor H. Kwakernaak.

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space equals two, the only systems for which the problem is robustly solvable are those that are state-equivalent to a linear system. As an example, it follows that there does not exist any $T^* > 0$ such that, for every $T \in (0, T^*)$ one can successfully apply the observer linearization design methodology to a sampled-data representation $\Sigma_d(T)$ of the system (a simple pendulum)

$$\Sigma: \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin(x_1) \\ y &= x_1 \end{aligned} \tag{1.1}$$

even though the methodology can be applied to (1.1), yielding an observer in differential equation form that can in turn be *approximately* discretized and then implemented digitally. The implications are that, if one wishes to apply this methodology, it is important to start with a continuous-time model of the system, or one must seek an *approximate* solution with respect to the sampling time, possibly along the lines of Lee *et al.* (1988).

The complexity of obtaining a precise characterization for the robust solvability of the observer linearization problem for higher dimensional systems is prohibitive. However, our development will make it clear that robust solvability with respect to time-sampling is a very stringent requirement, and hence the implications discussed in the preceding paragraph are valid.

Our results will be stated for exact sampled-data representations of a system. Of course, for a nonlinear system, this is usually not calculable. There are two good reasons, however, for working with such representations. Firstly, there are many degrees of approximation and we avoid having to choose amongst them. More importantly, any reasonable approximation technique should converge to the exact sampled-data representation as more and more terms are taken in the approximation. Our results imply that increasing the accuracy of the approximation does not get one closer to meeting the solvability conditions of the (perfect) observer linearization problem because they fail for the exact sampled-data representation. Instead, a solution to the *approximate* observer linearization problem should be sought.

In the remainder of the introduction we recall the few concepts from advanced calculus (Boothby, 1975) that will be used repeatedly throughout the paper. If $h: R^n \to R$ is a $C^\infty$-function, then $dh$ will denote its *differential*; that is $dh(x) = (\partial h(x)/\partial x_1, \ldots, \partial h(x)/\partial x_n)$, or, equivalently, $dh(x) = \sum_{i=1}^{n} (\partial h(x)/\partial x_i)\, dx_i$. If $f: R^n \to R^n$ is a $C^\infty$-vector field, that is, the right-hand-side of a differential equation defined on $R^n$, then the directional derivative of $h$ along $f$ is $L_f h(x) = \sum_{i=1}^{n} (\partial h(x)/\partial x_i) f_i(x)$, where $f(x) = (f_1(x), \ldots, f_n(x))'$. Note that $L_f h: R^n \to R$ and, hence, higher order directional derivatives can be defined recursively as $L_f^{k+1} h = L_f(L_f^k h)$. At times, in order to avoid putting $f$ as a subscript, the notation $\langle dh, f \rangle$ is used; that is, $\langle dh, f \rangle(x) = L_f h(x)$. This notation also symbolizes the "duality pairing" that exists between differentials and vector fields. If $g: R^n \to R^n$ is another $C^\infty$-vector field, the *Lie bracket* of $f$ with $g$ is the new vector field

$$[f, g](x) = \frac{\partial g(x)}{\partial x} f(x) - \frac{\partial f(x)}{\partial x} g(x).$$

In order that higher order Lie brackets of $f$ with $g$ can be defined conveniently, one also uses the "operator notation" $ad_f g(x) = [f, g](x)$ and defines, recursively, $ad_f^{k+1} g(x) = [f, ad_f^k g](x)$. Finally, it is straightforward to show that the Lie bracket and the directional derivative are related by $L_{[f,g]} h(x) = L_f L_g h(x) - L_g L_f h(x)$.

## 2. BACKGROUND: SCALAR OUTPUT CONTINUOUS-TIME SYSTEMS

Consider an uncontrolled dynamics

$$\dot{x} = Ax, \quad x \in R^n, \tag{2.1a}$$

with observations

$$y = Cx, \quad y \in R. \tag{2.1b}$$

As is well-known, if $(A, C)$ is an observable pair, then one can choose a vector $G$ and set up a differential equation for an approximation $\hat{x}(t)$ of $x(t)$,

$$\dot{\hat{x}} = A\hat{x} - G(y - C\hat{x}), \tag{2.2}$$

such that, upon defining $e = x - \hat{x}$, $\dot{e} = (A + GC)e$ is asymptotically stable. Consequently, $\hat{x}(t) \to x(t)$ as $t \to \infty$ and (2.2) constitutes an observer for (2.1).

Now, suppose we are trying to observe the state of a nonlinear system;

$$\dot{x} = f(x), \tag{2.3a}$$

$$y = h(x). \tag{2.3b}$$

In general, this is a difficult task. However, it is conceivable [Krener and Isidori (1983); for a physical example Kantor (1988)] that the system (2.3) is the result of applying nonlinear output injection and a change of coordinates to a linear system (2.1). When this is the case, then after a nonlinear change of coordinates, the system would have the form

$$\dot{z} = Az + \varphi(y) \qquad y = Cz. \tag{2.4}$$

Then, given $\varphi(y)$ and $z = \Phi(x)$, we can construct an observer for (2.3) almost as easily as for (2.1). Indeed, let the approximation $\hat{z}$ satisfy

$$\dot{\hat{z}} = A\hat{z} - G(y - C\hat{z}) + \varphi(y). \tag{2.5}$$

Then, the error $e = z - \hat{z}$ satisfies

$$\dot{e} = (A + GC)e, \tag{2.6}$$

as before, and it follows that $\hat{x}(t) := \Phi^{-1}(\hat{z}(t)) \to x(t)$ as $t \to \infty$.

We now summarize the results of Krener and Isidori (1983) characterizing those systems obtainable from a linear system by output injection and a change of coordinates. Let $\Sigma$ denote the system

$$\Sigma: \quad \begin{aligned} \dot{x} &= f(x) \\ y &= h(x), \end{aligned} \tag{2.7}$$

where $x(t) \in R^n$, $y(t) \in R$, and $f$ and $h$ are analytic functions of $x$. Suppose that $x^0$ is an equilibrium point of $\Sigma$. Then, $\Sigma$ is said to satisfy the *observability condition* if the one forms $dh, \ldots, dL_f^{n-1}h$ are linearly independent at $x^0$. $\Sigma$ is said to be *locally state-equivalent to a linear system* if there exists an open neighborhood $U$ of $x^0$ and a change of coordinates $z = \Phi(x)$, with $\Phi(x^0) = 0$, defined on $U$ such that, in the new coordinates, $\Sigma$ has the form

$$\Sigma': \quad \begin{aligned} \dot{z} &= Az \\ y &= Cz, \end{aligned} \tag{2.8}$$

where the pair $(A, C)$ is assumed observable. Finally, $\Sigma$ is *locally state-equivalent to a linear system with (nonlinear) output injection* if the above holds with $\Sigma'$ replaced by

$$\Sigma'': \quad \begin{aligned} \dot{z} &= Az + \varphi(y) \\ y &= Cz, \end{aligned} \tag{2.9}$$

where $\varphi$ is an analytic function of $y$ such that $\varphi(0) = 0$.

The two main results of Krener and Isidori (1983) are:

*Theorem* 1. The nonlinear system $\Sigma$ is locally state-equivalent to a linear system under a change of state coordinates $z = \Phi(x)$ where $\Phi(x^0) = 0$ if, and only if, $\Sigma$ satisfies the observability condition, $f(x^0) = 0$, $h(x^0) = 0$ and $dL_f^n h$ is an $R$-linear combination of $dL_f^k h$ for $k = 0, 1, \ldots, n - 1$.

*Theorem* 2. The nonlinear system $\Sigma$ is locally state-equivalent to a linear system with (nonlinear) output injection under a change of state coordinates $z = \Phi(x)$ where $\Phi(x^0) = 0$ if, and only if, $\Sigma$ satisfies the observability condition, $h(x^0) = 0$, $f(x^0) = 0$ and the unique vector field $g(x)$ defined by

$$L_g L_f^k h = \begin{cases} 0, & 0 \le k \le n-2 \\ 1, & k = n-1 \end{cases} \tag{2.10}$$

satisfies $[ad_f^i g, ad_f^j g] = 0$ for all $0 \le i,\ j \le n - 1$, or equivalently, $[g, ad_f^i g] = 0$ for all $0 \le i \le 2n - 3$.

*Remark* 1. (a) In Krener and Isidori (1983), it is shown that the vector field $g$ defined by (2.10) satisfies $[ad_f^i g, ad_f^j g] = 0$ for all $0 \le i, j \le n$, if, and only if, $\Sigma$ is locally state-equivalent to a linear system under a change of state coordinates. It is also shown that this condition is equivalent to the condition, $[g, ad_f^k g] = 0$ for all $0 \le k \le 2n - 1$.

(b) If $(A, C)$ is only detectable, then conditions guaranteeing the existence of the coordinate transformation $z = \Phi(x)$ are not known; such an extension would be nontrivial.

## 3. OBSERVER LINEARIZATION FOR DISCRETE-TIME SYSTEMS

Before we can consider the effects of time sampling on the observer linearization problem, we must obtain criteria for its solvability in the case of discrete-time nonlinear systems. The development largely parallels that of Krener and Isidori (1983); the proofs will be accordingly abridged.

In the following, we use the convention that, if $h$ and $F$ are two functions, $h$ being real valued, $F: R^n \to R^n$, then $dh \circ F := d(h \circ F)$; that is, composition takes precedence over the differential. Also, $F^k$ equals $F$ composed with itself $k$ times and $F^{-k} = (F^{-1})^k$ whenever $F$ is invertible.

Let $\Sigma_d$ denote the system

$$\Sigma_d: \quad \begin{aligned} x(k+1) &= F(x(k)) \\ y(k) &= h(x(k)), \end{aligned} \tag{3.1}$$

where $x(k) \in R^n$, $y(k) \in R$, $F$ and $h$ are analytic functions of $x$, and $F$ is a local diffeomorphism. Suppose that $x^0$ is an equilibrium point of (3.1); that is, $F(x^0) = x^0$. Then $\Sigma_d$ is said to satisfy the *observability condition* if the one forms $dh(x)$, $dh \circ F(x), \ldots, dh \circ F^{n-1}(x)$ are linearly independent at $x^0$. $\Sigma_d$ is said to be *locally state-equivalent to a linear system* if there exists an open neighborhood $U$ of $x^0$ and a change of coordinates $z = \Phi(x)$ with $\Phi(x^0) = 0$ defined on $U$ such that in the new coordinates $\Sigma_d$ has the form

$$\Sigma_d': \begin{array}{l} z(k+1) = Az(k) \\ y(k) = Cz(k), \end{array} \tag{3.2}$$

where the pair $(A, C)$ is observable. Finally, $\Sigma_d$ is *locally state-equivalent to a linear system with (nonlinear) output injection* if the above holds with $\Sigma_d'$ replaced by

$$\Sigma_d'': \begin{array}{l} z(k+1) = Az(k) + \varphi(y(k)) \\ y(k) = Cz(k), \end{array} \tag{3.3}$$

where $\varphi$ is an analytic function of $y$ such that $\varphi(0) = 0$. We note that if (3.1) satisfies the observability condition, then it is always possible to apply (in any given coordinate system) a linear output injection which makes $F$ a local diffeomorphism in a neighborhood of an equilibrium point. Hence, our assumption on $F$ was without loss of generality. Of course, as in the case of a continuous-time system, the interest of (3.3) is that one may locally construct an exact observer by

$$\begin{aligned} \hat{z}(k+1) = A\hat{z}(k) - G(y(k) - C\hat{z}(k)) \\ + \varphi(y(k)) \end{aligned} \tag{3.4}$$

and if $G$ is selected appropriately, then $\hat{x}_k := \Phi^{-1}(\hat{z}_k) \to x(k)$, as $k$ tends toward infinity.

*Theorem* 3. The nonlinear system $\Sigma_d$ is locally state-equivalent to a linear system of the form $\Sigma_d'$ under a change of state coordinates $z = \Phi(x)$ where $\Phi(x^0) = 0$ if, and only if, $\Sigma_d$ satisfies the observability condition, $F(x^0) = x^0$, $h(x^0) = 0$ and $dh \circ F^n(x)$ is an $R$-linear combination of $dh \circ F^k(x)$ for $k = 0, 1, \ldots, n-1$.

*Proof.* The proof is the same as in continuous-time case; see Krener and Isidori (1983) □

In the sequel, we define, following Jakubczyk and Sontag (1990), for a given vector field $g'(x)$, $Ad_F g'(x) := (F_* g')(F^{-1}(x))$, and $Ad_F^{k+1} g'(x) := (F_* Ad_F^k g')(F^{-1}(x))$. It follows easily that $Ad_F^k g'(x) = (F_*^k g')(F^{-k}(x)) = Ad_{F^k} g'(x)$. This is also valid for $k < 0$.

*Theorem* 4. The nonlinear system $\Sigma_d$ is locally state-equivalent to a linear system with output injection under a change of state coordinates $z = \Phi(x)$ where $\Phi(x^0) = 0$ if, and only if, $\Sigma$ satisfies the observability condition, $F(x^0) = x^0$, $h(x^0) = 0$ and there exists an open neighborhood $U$ of $x^0$ such that the unique vector field $g'(x)$ defined on $U$ by

$$L_{g'} h \circ F^k = \begin{cases} 0; & 0 \le k \le n-2 \\ 1; & k = n-1 \end{cases} \tag{3.5}$$

satisfies

$$[g', Ad_F^i g'] = 0 \quad \text{for} \quad -(n-1) \le i \le n-1. \tag{3.6}$$

In order to prove the above Theorem, we need the following lemmas.

*Lemma* 1. The condition (3.5) is equivalent to

$$L_{Ad_F^k g'} h = \begin{cases} 0; & 0 \le k \le n-2 \\ 1; & k = n-1. \end{cases} \tag{3.7}$$

*Proof*

$$\begin{aligned} (L_{g'} h \circ F)(x) &= \frac{\partial h \circ F}{\partial x}\bigg|_x g'(x) \\ &= \frac{\partial h}{\partial x}\bigg|_{F(x)} \frac{\partial F}{\partial x}\bigg|_x g'(x) \\ &= \frac{\partial h}{\partial x}\bigg|_{F(x)} F_* g'(F^{-1}(F(x)) \\ &= (L_{Ad_F g'} h)(F(x)), \end{aligned}$$

and therefore, $L_{Ad_F g'} h = 0$. In the same way, it follows that $(L_{g'} h \circ F^k)(x) = (L_{Ad_F^k g'} h)(F^k(x))$, proving the Lemma.

*Lemma* 2. $[Ad_F^i g', Ad_F^j g'] = 0$ for $0 \le i, j \le n-1$, if, and only if, $[g', Ad_F^k g'] = 0$ for $-(n-1) \le k \le n-1$.

*Proof.* (Necessity) We only need to prove the result for $k < 0$. Since $F^k$ is a local diffeomorphism,

$$\begin{aligned} 0 &= (F_*^k [g', Ad_F^k g'])(F^{-k}(z)) \\ &= [(F_*^k g')(F^{-k}(z)), (F_*^k Ad_F^k g')(F^{-k}(z))] \\ &= [Ad_F^k g', g'](z). \end{aligned}$$

(Sufficiency) The same argument works. □

*Lemma* 3. The condition

$$L_{Ad_F^k g'} h = \begin{cases} 0; & 0 \le k \le n-2 \\ 1; & k = n-1 \end{cases} \tag{3.8}$$

coupled with the observability condition, implies that the set of vectors $\{g'(x^0), \ldots, Ad_F^{n-1} g'(x^0)\}$ spans $R^n$.

*Proof.* We note the following identity holds for $0 \le i, j \le n-1$

$$\langle dh \circ F^i, Ad_F^j g' \rangle (x^0) = \langle dh, Ad_F^{i+j} g' \rangle (x^0). \tag{3.9}$$

See Appendix A for the proof. Then (3.9) yields

$$\langle dh \circ F^i, Ad_F^j g' \rangle (x^0) = \begin{cases} 0; & 0 \le i+j \le n-2 \\ 1; & i+j = n-1. \end{cases}$$

Therefore,

$$\begin{bmatrix} dh \\ \vdots \\ dh\circ F^{n-1} \end{bmatrix} [g', \ldots, Ad_F^{n-1}g'](x^0)$$
$$= \begin{bmatrix} \langle dh, g'\rangle(x^0) & \cdots & \langle dh, Ad_F^{n-1}g'\rangle(x^0) \\ \vdots & & \vdots \\ \langle dh\circ F^{n-1}, g'\rangle(x^0) & \cdots & \langle dh\circ F^{n-1}, Ad_F^{n-1}g'\rangle(x^0) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & 1 & * \\ 0 & 1 & & \vdots \\ 1 & * & & * \end{bmatrix}.$$

This final matrix has rank $n$, and thus $\{g'(x^0), \ldots, Ad_F^{n-1}g'(x^0)\}$ is a linearly independent set. □

We return to the proof of Theorem 4 (Sufficiency). By Lemmas 1 and 3, $\{g'(x^0), \ldots, Ad_F^{n-1}g'(x^0)\}$ spans $R^n$, and by Lemma 2, $[Ad_{FG}^i g', Ad_{FG}^j g'] = 0$, $0 \le i, j \le n-1$. Hence, the Frobenius Theorem (Boothby, 1975; Isidori, 1985) guarantees the existence of local coordinates $z = \Phi(x)$ such that $\Phi(x^0) = 0$ and for $k = 0, 1, \ldots, n-1$, $(\partial/\partial z_{n-k}) = Ad_{FG}^k g'$. In these coordinates, for $0 \le k \le n-2$

$$F_*\left\{\frac{\partial}{\partial z_{n-k}}\right\}(F^{-1}(z)) = F_*\{Ad_{FG}^k g'\}(F^{-1}(z)) = Ad_F^{k+1}g'(z) = \frac{\partial}{\partial z_{n-k-1}},$$

and, for $0 \le k \le n-1$

$$F_*\left\{\frac{\partial}{\partial z_{n-k}}\right\}(F^{-1}(z)) = \sum_{i=1}^{n} \frac{\partial F_i}{\partial z_{n-k}}\bigg|_{F^{-1}(z)} \frac{\partial}{\partial z_i}.$$

Thus, for $0 \le k \le n-2$,

$$\frac{\partial F_i}{\partial z_{n-k}} = \begin{cases} 1; & i = n-k-1 \\ 0; & \text{otherwise.} \end{cases}$$

Also, from Lemma 1

$$\frac{\partial h(z)}{\partial z_{n-k}} = L_{Ad_{FG}^k g'} h(z) = \begin{cases} 0; & 0 \le k < n-1 \\ 1; & k = n-1. \end{cases}$$

Thus

$$y = Cz = (1, 0, \ldots, 0)\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = z_1,$$

and

$$F(z) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \cdots & & & 1 \\ 0 & \cdots & & & 0 \end{pmatrix} \times \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} + \begin{pmatrix} \varphi_1(z_1) \\ \varphi_2(z_1) \\ \vdots \\ \varphi_n(z_1) \end{pmatrix} = Az + \varphi(y).$$

(Necessity) It is straightforward and left to the reader. □

We remark that the constant 1 used in Theorem 4, Equation (3.5), can be replaced by any nonzero constant $\epsilon$ and the same results hold; this is because the vector field $g'$ is just scaled by $1/\epsilon$.

## 4. THE EFFECTS OF SAMPLING ON OBSERVER LINEARIZATION

We are now in a position to investigate the effects of time-sampling on the solvability of the observer linearization problem of Sections 2 and 3. Let

$$\Sigma_d(T): \begin{array}{l} x_{k+1} = F_T(x_k) \\ y_k = h(x_k) \end{array} \tag{4.1}$$

be the result of time-sampling the system (2.7); that is, $x_k := x(kT)$, where $x(t)$ is the solution of (2.7), and $y_k := y(kT)$. Of course, (4.1) may only be defined for $T$ sufficiently small unless the vector field $f(x)$ in (2.7) is complete (Boothby, 1975). To begin with, note that if $x^0$ is an equilibrium point of (2.7), then it is also an equilibrium point of (4.1). Next, (2.7) satisfies the observability condition at $x^0$ if, and only if, (4.1) satisfies the observability condition at $x^0$ for all $T$ sufficiently small, but not equal to zero. This is because $dL_f^k h(x^0) = \bar{C}\bar{A}^k$, where $\bar{A} = \partial f/\partial x(x^0)$ and $\bar{C} = \partial h/\partial x(x^0)$, and $dh\circ F_T^k(x^0) = \bar{C}\exp(k\bar{A}T)$, as a simple computation shows; hence, since observability is preserved for linear systems for sufficiently small sampling times (Sontag, 1984), the result follows. Finally, we recall the following formula (Isidori, 1985; Varadarajan, 1984) which makes explicit the relation between the sampled system (4.1) and the underlying continuous-time system (2.7):

for sufficiently small $T$, $h\circ F_T(x)$

$$= \sum_{k=0}^{\infty} L_f^k h(x)\frac{T^k}{k!}. \tag{4.2}$$

In this section, we always work in a neighborhood of a given equilibrium point $x^0$. Our first result addresses the property of being state-equivalent to a linear system with outputs. See Theorem 3.1 in Arapostathis *et al.* (1989) for the corresponding result for systems with inputs, but no outputs.

*Theorem* 5. The following two statements are equivalent.
(a) The continuous-time system $\Sigma$ is locally state-equivalent to a linear system.
(b) There exists $T^* > 0$ such that for each $T \in (0, T^*)$, the sampled data system $\Sigma_d(T)$ is locally state-equivalent to a linear system.

*Proof.*
(a)⇒(b) is straightforward and left to the reader.
(b)⇒(a) When $n$, the dimension of the state space, equals 1, the proof is easy and also left to the reader. Now, in order to simplify the notation, we assume that $n$ equals 2; the same proof works for $n \geq 3$. Since $\Sigma_d(T)$ satisfying the observability condition for an open set of sampling times implies that $\Sigma$ satisfies the observability condition, it follows that $dh$ and $dL_fh$ locally span $T^*R^2$, the cotangent bundle of $R^2$. Therefore, there locally exist analytic functions $\alpha$ and $\beta$ such that

$$dh \circ F_T(x) = \alpha(x, T)\, dh(x) + \beta(x, T)\, dL_fh(x). \tag{4.3}$$

Expanding $\alpha$ and $\beta$ in a Maclaurin series with respect to $T$,

$$\begin{aligned}\alpha(x, T) &=: a_0(x) + a_1(x)T + \frac{T^2}{2!}a_2(x) + \cdots + \frac{T^k}{k!}a_k(x) + \cdots,\\ \beta(x, T) &=: b_0(x) + b_1(x)T + \frac{T^2}{2!}b_2(x) + \cdots + \frac{T^k}{k!}b_k(x) + \cdots.\end{aligned} \tag{4.4}$$

From (4.2), one obtains, for each $k \geq 0$,

$$\frac{d^k}{dT^k} dh \circ F_T(x)\,|_{T=0} = dL_f^kh(x). \tag{4.5}$$

This, combined with (4.4), yields

$$dL_f^ih(x) = a_i(x)\, dh(x) + b_i(x)\, dL_fh(x), \tag{4.6}$$

from which one deduces that $a_0(x) \equiv 1$, $a_1(x) \equiv 0$, $b_0(x) \equiv 0$, and $b_1(x) \equiv 1$. The goal is simply to show that $a_2(x)$ and $b_2(x)$ are actually constants (recall Theorem 1). Combining now hypothesis (b) with Theorem 3, one deduces that

$$dh \circ F_T^2(x) = c_0(T)\, dh(x) + c_1(T)\, dh \circ F_T(x) \tag{4.7}$$

where $c_0$, $c_1$ are analytic functions of $T$. Substituting (4.3) into (4.7) gives

$$\begin{aligned}dh \circ F_T^2(x) = \{c_0(T) + c_1(T)\alpha(x, T)\}\, dh(x)\\ + c_1(T)\beta(x, T)\, dL_fh(x).\end{aligned} \tag{4.8}$$

Since $dh \circ F_T^2(x) = dh \circ F_{2T}(x)$, (4.3) and (4.7) yield

$$c_1(T)\beta(x, T) = \beta(x, 2T), \tag{4.9}$$

$$c_0(T) = -c_1(T)\alpha(x, T) + \alpha(x, 2T). \tag{4.10}$$

Therefore, for all $k \geq 0$,

$$\frac{d^k}{dT^k}[c_1(T)\beta(x, T)]\,|_{T=0} = 2^kb_k(x) \tag{4.11}$$

$$\frac{d^k}{dT^k}[c_0(T) + c_1(T)\alpha(x, T)]\,|_{T=0} = 2^ka_k(x). \tag{4.12}$$

Recalling that $a_0(x) \equiv 1$, $a_1(x) \equiv 0$, $b_0(x) \equiv 0$ and $b_1(x) \equiv 1$, a straightforward calculation then gives that $a_2$ and $b_2$ are independent of $x$. This completes the proof. □

Motivated by Theorem 5, we introduce the following definition.

*Definition* 1. $\Sigma$ is said to be *sampled-data observer linearizable* if there exists a $T^* > 0$ such that for all $T \in (0, T^*)$, the sampled-data system $\Sigma_d(T)$ is locally state-equivalent to a linear system with output injection (and hence the observer error dynamics can be linearized).

Then, from the results for discrete-time systems of Section 3, the following lemma is immediate.

*Lemma* 4. $\Sigma$ is sampled-data observer linearizable if, and only if, $\Sigma$ satisfies the observability condition, and there exist a $T^* > 0$, such that the unique auxiliary vector field $g'(x, T)$ defined by

$$L_{g'}h \circ F_T^k = \begin{cases} 0; & 0 \leq k \leq n-2 \\ T^{n-1}; & k = n-1 \end{cases} \quad \text{for all } T \in (0, T^*), \tag{4.13}$$

satisfies

$$[g', Ad_{F_T}^i g'] = 0 \quad \text{for} \quad -(n-1) \leq i \leq n-1 \quad \text{and for all } T \in (0, T^*). \tag{4.14}$$

*Proof.* The proof is parallel to that of Theorem 4, and hence is omitted. □

The vector field $g'$ in (4.13) is obtained by solving the set of linear equations

$$\begin{bmatrix} dh(x) \\ dh \circ F_T(x) \\ \vdots \\ dh \circ F_T^{n-1}(x) \end{bmatrix} g'(x, T) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ T^{n-1} \end{bmatrix}. \tag{4.15}$$

Since the matrix on the left hand side has rank $n$ for $0 < T < T^*$, and is an analytic function of $x$ for $x$ near $x^0$, and of $T$ for $0 < T < T^*$, it follows that $g'(x, T)$ is an analytic function of $x$ for $x$ near $x^0$, and of $T$ for $0 < T < T^*$. In the following, we actually need $g'$ to be analytic in $T$ for $T$ in an *open neighborhood of the origin* and we now show that (4.15) admits such a solution. When $T^{n-1}$ is replaced by 1, for example as in equation (3.5), then this is impossible.

To begin with, choose $T^*$ sufficiently small so that $F_T^k(x)$ exists for all $-T^* < T < T^*$, for all $x$ in an open neighborhood of $x^0$, and for all $1 \le k \le n$. Recall from (4.2) that

$$\mathrm{d}h \circ F_T = \sum_{i=0}^{\infty} \frac{T^i}{i!} \mathrm{d}L_f^i h. \tag{4.16}$$

Because $\Sigma$ satisfies the observability condition, $\mathrm{d}L_f^k h$ $(k \ge n)$ can be represented as a linear combination of $\mathrm{d}h, \ldots, \mathrm{d}L_f^{n-1}h$. Thus, let

$$\mathrm{d}L_f^k h := \mathrm{d}_0^k \,\mathrm{d}h + \cdots + \mathrm{d}_{n-1}^k \,\mathrm{d}L_f^{n-1}h \tag{4.17}$$

where $\mathrm{d}_0^k(x), \ldots, \mathrm{d}_{n-1}^k(x)$ are appropriate analytic functions of $x$. Then,

$$\mathrm{d}h \circ F_T = \sum_{i=0}^{n-1} \gamma_i(x, T)\, \mathrm{d}L_f^i h, \tag{4.18}$$

where

$$\gamma_i(x, T) = \frac{T^i}{i!} + \sum_{m=n}^{\infty} \mathrm{d}_i^m(x) \frac{T^m}{m!},$$

and it follows that

$$\mathrm{d}h \circ F_T^k = \mathrm{d}h \circ F_{kT} = \sum_{i=0}^{n-1} \gamma_i(x, kT)\, \mathrm{d}L_f^i h. \tag{4.19}$$

Substituting (4.18) and (4.19) into equation (4.15), we have the following expression in local coordinates,

$$RHg'(x, T) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ T^{n-1} \end{bmatrix},$$

where

$$R = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \gamma_0(x, T) & \gamma_1(x, T) & \cdots & \gamma_{n-1}(x, T) \\ \vdots & \vdots & & \vdots \\ \gamma_0(x, (n-1)T) & \gamma_1(x, (n-1)T) & \cdots & \gamma_{n-1}(x, (n-1)T) \end{bmatrix}$$

and

$$H = \begin{bmatrix} \dfrac{\partial h}{\partial x_1} & \dfrac{\partial h}{\partial x_2} & \cdots & \dfrac{\partial h}{\partial x_n} \\ \dfrac{\partial L_f h}{\partial x_1} & \dfrac{\partial L_f h}{\partial x_2} & \cdots & \dfrac{\partial L_f h}{\partial x_n} \\ \vdots & & & \vdots \\ \dfrac{\partial L_f^{n-1} h}{\partial x_1} & \dfrac{\partial L_f^{n-1} h}{\partial x_2} & \cdots & \dfrac{\partial L_f^{n-1} h}{\partial x_n} \end{bmatrix}.$$

The matrix $H$ has rank $n$ near $x^0$ due to the observability condition, but $R$ only has rank $n$ for $0 < |T| < T^*$, for $T^*$ sufficiently small. Define

$$\bar{g}(x, T) := Hg'(x, T) \tag{4.20}$$

so that

$$R(x, T)\bar{g}(x, T) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ T^{n-1} \end{bmatrix}. \tag{4.21}$$

The goal now is to show that (4.21) admits an analytic solution $\bar{g}(x, T)$ for $T$ in an open neighborhood of 0. By Cramer's rule (Hoffman and Kunze, 1971),

$$\bar{g}(x, T) := (\det R)^{-1} \begin{bmatrix} 0 \\ T^{n-1}(-1)^{2+n} \det R(n \mid 2) \\ \vdots \\ T^{n-1}(-1)^{n+n} \det R(n \mid n) \end{bmatrix},$$

where $\det R(n \mid i)$ is the $(n, i)$ cofactor of $R$; i.e. the determinant of $R$ when the $n$th row and $i$th column are deleted. This represents each component of $\bar{g}$ as a meromorphic function (the ratio of two analytic functions). However, from direct calculations, the lowest order in $T$ of $\det R$ is $(n-1)n/2$, and the lowest possible order in $T$ of $\det R(n \mid i)$ $(i \ge 2)$ is $(n-1)n/2 - (i-1)$ [see Appendix B]. Thus, each term $T^{n-1}(-1)^{i+n} \det R(n \mid i)$ has the lowest order in $T$ not less than that of $\det R$, and therefore $\bar{g}(x, T)$ is analytic in $T$, for $T$ sufficiently small. □

Using the above results, we can now prove the following theorem which gives an "upper bound" on the class of continuous-time systems for which the observer linearization property is robust with respect to time-sampling. We will show later that the class of system is actually much smaller than proved here. See Theorem 4.1 in Arapostathis *et al.* (1989) for the corresponding result on the feedback linearization problem.

*Theorem* 6. If the system $\Sigma$ is sampled data observer linearizable, then $\Sigma$ is locally state-equivalent to a linear system with output injection.

*Proof.* If $\Sigma$ is sampled data observer linearizable with output injection, then from the above results, $\Sigma$ satisfies the observability condition and we have a $T^* > 0$ and an analytic auxiliary vector field $g'(x, T)$ which satisfies

$$\begin{bmatrix} dh(x) \\ dh \circ F_T(x) \\ \vdots \\ dh \circ F_T^{n-1}(x) \end{bmatrix} g'(x, T) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ T^{n-1} \end{bmatrix} \tag{4.22}$$

and the Lie bracket condition $[g', Ad_{F_T}^k g'] = 0$ for $-(n-1) \le k \le n-1$ and for all $-T^* < T < T^*$. Since $g'(x, T)$ is analytic in $T$, for $T$ near 0, we can expand $g'$ as

$$g'(x, T) := \sum_{i=0}^{\infty} g_i(x) T^i, \tag{4.23}$$

where the $g_i(x)$'s are analytic functions of $x$. Substitute (4.23) and (4.16) into (4.22). Then, we have

$$\langle dh \circ F_T^k(x), g'(x, T) \rangle = \sum_{i=0}^{\infty} \left\{ \sum_{j=0}^{i} \frac{(k)^j}{j!} L_{g_{(i-j)}} L_f^j h(x) \right\} T^i \tag{4.24}$$

which yields the set of equations

$$\begin{bmatrix} dh \\ dh \circ F_T \\ \vdots \\ dh \circ F_T^{n-1} \end{bmatrix} g'(\cdot, T) = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [L_{g_0} h] + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \\ 1 & (n-1) \end{bmatrix} \times \begin{bmatrix} L_{g_1} h \\ L_{g_0} L_f h \end{bmatrix} T + \cdots + \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1/(n-1)! \\ 1 & 2 & \cdots & \frac{2^{n-1}}{(n-1)!} \\ \vdots & & & \\ 1 & (n-1) & \cdots & \frac{(n-1)^{n-1}}{(n-1)!} \end{bmatrix}$$

$$\times \begin{bmatrix} L_{g_{n-1}} h \\ L_{g_{n-2}} L_f h \\ L_{g_{n-3}} L_f^2 h \\ \vdots \\ L_{g_0} L_f^{n-1} h \end{bmatrix} T^{n-1} + \cdots = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ T^{n-1} \end{bmatrix}. \tag{4.25}$$

This is true for all $T$ in an open interval about zero. Thus, it follows by comparing both sides of (4.25), that $L_{g_0} h = L_{g_0} L_f h = \cdots = L_{g_0} L_f^{n-2} h = 0$, and $L_{g_0} L_f^{n-1} h = 1$ (see Appendix C). That is, we have

$$L_{g_0} L_f^k h = \begin{cases} 0; & 0 \le k \le n-2 \\ 1; & k = n-1. \end{cases} \tag{4.26}$$

Now, we want to show, in view of Theorem 2, that $[g_0, ad_f^k g_0] = 0$ for $k = 0, 1, \ldots, 2n-3$. We recall first the Baker–Campbell–Hausdorff formula (Isidori, 1985; Varadarajan, 1984):

$$Ad_{F_T} g(x) := (F_T)_* g(F_T^{-1}(x)) = \sum_{i=0}^{\infty} ad_f^i g(x) \frac{(-T)^i}{i!}.$$

Thus

$$Ad_{F_{(kT)}} g = \sum_{i=0}^{\infty} ad_f^i g \frac{(-kT)^i}{i!} \quad (-(n-1) \le k \le n-1) \tag{4.27}$$

and, by substituting $g' = \sum_{i=0}^{\infty} g_i T^i$ into (4.27), we have

$$Ad_{F_{(kT)}} g' = \sum_{i=0}^{\infty} \left\{ \sum_{j=0}^{i} ad_f^j g_{(i-j)} \frac{(-k)^j}{j!} \right\} T^i. \tag{4.28}$$

Now if we substitute $g' = \sum_{i=0}^{\infty} g_i T^i$, and (4.28) into $[g', Ad_{F_{(kT)}} g']$, we have

$$[g', Ad_{F_{(kT)}} g'] = [g_0, g_0] + \sum_{i=1}^{\infty} \left\{ \sum_{j=0}^{i-1} \frac{(-k)^{(i-j)}}{(i-j)!} \times \left( \sum_{m=0}^{j} [g_m, ad_f^{i-j} g_{j-m}] \right) \right\} T^i. \tag{4.29}$$

Finally, we substitute (4.29) into

$$0 = \begin{pmatrix} [g', Ad_{F_T}^{-(n-1)} g'] \\ [g', Ad_{F_T}^{-(n-2)} g'] \\ \vdots \\ [g', Ad_{F_T}^{-1} g'] \\ [g', Ad_{F_T} g'] \\ \vdots \\ [g', Ad_{F_T}^{(n-2)} g'] \\ [g', Ad_{F_T}^{(n-1)} g'] \end{pmatrix},$$

to obtain the following matrix equation:

$$0 = b_0 + \sum_{i=1}^{\infty} (M_i \otimes I_{n \times n}) b_i T^i, \tag{4.30}$$

where $\otimes$ is the Kronecker product of matrices, the $M_i'$s are $(2n-2) \times i$ matrices and the $b_i'$s are $ni \times 1$ vectors such that:

$$b_0 := [g_0, g_0] = 0$$

$$M_i := \begin{bmatrix} (n-1)^i/i! & (n-1)^{i-1}/(i-1)! & \cdots & (n-1) \\ (n-2)^i/i! & (n-2)^{i-1}/(i-1)! & \cdots & (n-2) \\ \vdots & & & \\ 1/i! & 1/(i-1)! & \cdots & 1 \\ (-1)^i/i! & (-1)^{i-1}/(i-1)! & \cdots & -1 \\ \vdots & & & \\ (-(n-1))^i/i! & (-(n-1))^{i-1}/(i-1)! & \cdots & -(n-1) \end{bmatrix},$$

$$b_i := \begin{bmatrix} [g_0, ad_f^i g_0] \\ \vdots \\ \sum_{m=0}^{j-1} [g_m, ad_f^{i-j+1} g_{j-m-1}] \\ \vdots \\ \sum_{m=0}^{i-1} [g_m, ad_f g_{i-1-m}] \end{bmatrix} \leftarrow j\text{th place.}$$

It is a simple matter to show that for $1 \le i \le 2n-2$, the rank of $M_i$ equals $i$; see Appendix D. Hence, rank $M_i \otimes I_{n\times n} = ni$ which proves that $b_i = 0$ and therefore, $[g_0, ad_f^i g_0] = 0$ for $1 \le i \le 2n-2$. This completes the proof. □

Theorem 6 shows that the class of systems that are sampled-data observer linearizable is no larger than $\{\Sigma_d(T) : T > 0$ and $\Sigma$ is a continuous-time system that is linearizable with output injection.$\}$. When the dimension of the state space is one, this consists of all system such that $h(0) = 0$ and $\partial h/\partial x(0) \neq 0$. When the dimension of the state space is two, we can show that a system is sampled-data observer linearizable if, and only if, it is locally state equivalent to a linear system. The point is that, in proving Theorem 6, we only used part of the conditions coming from (4.25) and (4.30) and the additional conditions pose highly nongeneric constraints on the class of sampled-data observer linearizable systems. See Theorem 5.3 in Arapostathis *et al.* (1989) for the corresponding result on the feedback linearization problem.

*Theorem* 7. When the dimension of the state space equals two, a system is sampled-data observer linearizable if, and only if, it is locally state-equivalent to a linear system.

*Proof.* Sufficiency being obvious (see Theorem 5), we only prove the necessity. The proof of Theorem 6 already used the conditions:

$$L_{g_0}h = 0, \quad L_{g_0}L_f h = 1, \quad [g_0, ad_f g_0] = 0. \quad (4.31)$$

From (4.25), we obtain in addition:

$$L_{g_1}h = L_{g_2}h = 0 \quad (4.32)$$

$$L_{g_1}L_f h + 1/2 L_{g_0}L_f^2 h = 0 \quad (4.33)$$

$$L_{g_2}L_f h + 1/2 L_{g_1}L_f^2 h + 1/6 L_{g_0}L_f^3 h = 0. \quad (4.34)$$

From (4.30), we obtain the additional conditions,

$$[g_0, ad_f^2 g_0] = 0 \quad (4.35)$$

$$[g_0, ad_f g_1] + [g_1, ad_f g_0] = 0 \quad (4.36)$$

$$[g_0, ad_f g_2] + [g_1, ad_f g_1] + [g_2, ad_f g_0] + \tfrac{1}{6}[g_0, ad_f^3 g_0] = 0 \quad (4.37)$$

where in the last equation, the term $1/2([g_0, ad_f^2 g_1] + [g_1, ad_f^2 g_0])$ was eliminated because it is zero by the Jacobi identity. The goal is to use (4.32)–(4.37) to prove that $[g_0, a\, d_f^3 g_0] = 0$ (recall Remark 1). Equation (4.32) implies that $g_1(x) = \eta_1(x) g_0(x)$, and $g_2(x) = \eta_2(x) g_0(x)$ for some analytic functions $\eta_1(x)$, and $\eta_2(x)$. Equation (4.36) then yields that $L_{g_0}\eta_1 = 0$ and $L_{ad_f g_0}\eta_1 = 0$. Therefore, $\eta_1$ is constant and equation (4.33) in turn shows that $L_{g_0}L_f^2 h = -2\eta_1 = \text{constant}$. From $[g_0, ad_f g_0] = 0$, one obtains the useful fact that

$$0 = \langle dL_f^2 h, [g_0, ad_f g_0] \rangle = -L_{g_0}^2 L_f^3 h. \quad (4.38)$$

Similarly, (4.35) establishes

$$\begin{aligned} 0 &= dL_f^2 h, [g_0, ad_f^2 g_0] \rangle \\ &= L_{g_0}^2 L_f^4 h - 2L_{g_0}L_f L_{g_0} L_f^3 h. \end{aligned} \quad (4.39)$$

Using all of these facts, (4.37) then yields

$$[g_0, ad_f^3 g_0] = (2L_{g_0}L_f L_{g_0}L_f^3 h) g_0. \quad (4.40)$$

Therefore,

$$\langle dh, [g_0, ad_f^3 g_0] \rangle = 0 \quad (4.41a)$$

$$\langle dL_f h, [g_0, ad_f^3 g_0] \rangle = 2L_{g_0}L_f L_{g_0}L_f^3 h. \quad (4.41b)$$

However, a direct calculation yields

$$\langle dL_f h, [g_0, ad_f^3 g_0] \rangle = L_{g_0}L_f L_{g_0}L_f^3 h. \quad (4.42)$$

Equations (4.41) and (4.42) together imply that $[g_0, ad_f^3 g_0] = 0$, which completes the proof. □

*Remark* 2. In this section, several results (Theorem 5, Lemma 4, Theorem 6, and Theorem 7) have been shown for an open set of times of the form $(0, T^*)$. We will now extend the results to more general open sets.

Let $\bar{T}$ be the largest $T$ such that

$$\text{rank} \begin{bmatrix} \bar{C} \\ \bar{C}\exp \bar{A}T \\ \vdots \\ \bar{C}\exp \bar{A}(n-1)T \end{bmatrix} = n$$

for all $T \in (0, \bar{T})$; that is, the largest $T$ such that the linearization is observable. Recall that $dh \circ F_T^k(x^0) = \bar{C}\exp \bar{A}kT$. Let $\tilde{T}$ be any fixed $T$ such that $0 < \tilde{T} < \bar{T}$. Then, there exists a unique vector field $g'(x, T)$ and a simply connected open set $O'$ of $x^0 = 0$ such that

$$\begin{bmatrix} dh(x) \\ dh \circ F_T(x) \\ \vdots \\ dh \circ F_T^{n-1}(x) \end{bmatrix} g'(x, T) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ T^{n-1} \end{bmatrix}$$

$$\text{for all } x \in O', \quad -\tilde{T} < T < \tilde{T}.$$

This in turn implies that there exists a possibly smaller open set $O \subset O'$ such that,

$$[Ad^i_{F_T}g' Ad^j_{F_T}g'](x) \quad \text{for} \quad 0 \le i, j \le n-1, \qquad (4.41)$$

is analytic for $x \in O$, $-\bar{T} < T < \bar{T}$. Therefore, if there exists an open set of sampling times $(T^* - \epsilon/2, T^* + \epsilon/2) \subset (-\bar{T}, \bar{T})$ on which the functions (4.41) all vanish, then they must vanish on all of $(-\bar{T}, \bar{T})$ due to analyticity. This reduces the problem to the one studied at the beginning of the section. Note that our assumption on $\bar{T}$ means that we are not undersampling the system.

## 5. CONCLUSIONS

We have investigated the effects of time-sampling on the solvability conditions for the observer error linearization problem. Requiring that the problem be solvable for an open set of sampling times places very stringent requirements on the system. Indeed, when the dimension of the state space is two, one is reduced to those systems that are state-equivalent to a linear system. For higher dimensional systems, the complexity of the calculations required in order to prove such a precise characterization is prohibitive, but the general implications are quite clear: either one starts with a continuous-time plant model, carries out the methodology (if applicable) and then implements an *approximate* digital observer by using "rapid" sampling, or an *approximate* solution must be sought to the discrete-time version of the problem. The latter problem is of interest in and of itself; the results presented here highlight its importance for applications.

Our results were obtained in the context of systems having a single output and *no* input. This was done for simplicity of exposition and because the results with inputs still have to hold when input is set equal to zero. In any case, it can be shown that the same arguments straightforwardly extend to systems with a single input; see Xia and Gao (1989).

The literature on approximate solutions to the continuous-time and discrete-time feedback linearization problems is growing (Krener, 1984; Goldthwait and Hunt, 1987; Lee and Marcus, 1986). The design philosophy that has emerged from these references is that feedback linearizable systems should be viewed as a richer class of approximations to a general nonlinear control system than is the class of linear constant coefficient systems. Still in the context of feedback linearization, methods for increasing the accuracy of a digital implementation of the continuous-time feedback law have been proposed (Lee *et al.*, 1988; Monaco *et al.*, 1986) and a multi-rate method has been studied (Grizzle and Kokotovic, 1988). Similar progress on observer linearization has been reported (Krener, 1986, 1989; Krener *et al.*, 1987; Phelps and Krener, 1988; Grizzle and Moraal, 1989).

## REFERENCES


Arapostathis, A., B. Jakubczyk, H.-G. Lee, S. I. Marcus and E. S. Sontag (1989). The effect of sampling on linear equivalence and feedback linearization. *Syst. Control Lett.*, **13**, 373–381.

Bestle, D. and M. Zeitz (1983). Canonical form observer design for nonlinear time variable systems. *Int. J. Contr.*, **38**, 419–431.

Boothby, W. M. (1975). *An Introduction to Differentiable Manifolds and Riemannian Geometry*. Academic Press, New York.

Goldthwait, R. G. and L. R. Hunt (1987). Nonlinear system approximations. *Proc. 26th IEEE Conf. on Decision and Control*, Los Angeles, CA., 1752–1756.

Grizzle, J. W. (1986). Feedback Linearization of Discrete-time Systems. In *Lecture Notes in Control and Information Sciences*, **83**, 273–281.

Grizzle, J. W. and P. V. Kokotovic (1988). Feedback linearization of sampled-data systems. *IEEE Trans. Aut. Control*, **AC-33**, 857–859.

Grizzle, J. W. and P. Moraal (1990). On observers for smooth nonlinear digital systems. In *Lecture Notes in Control and Information Sciences*, **144**, pp. 401–412.

Hoffman, K. and R. Kunze (1971). *Linear Algebra*, Prentice-Hall, Englewood Cliffs, NJ.

Isidori, A. (1985). *Nonlinear Control Systems: An Introduction*, (*Lecture Notes in Control and Information Sciences Vol.* 72). Springer-Verlag, Berlin.

Jakubczyk, B. and E. D. Sontag (1990). Controllability of nonlinear discrete time systems: A Lie algebraic approach. *SIAM J. Control*, **28**, 1–33.

Kantor, J. C. (1988). A rapprochement of feedforward and feedback linearization with process control practice. *Proc. ACC*, 1552–1557.

Krener, A. J. (1984). Approximate linearization by state feedback and coordinate change. *Syst. Control Lett.*, **5**, 181–185.

Krener, A. J. (1986). Normal forms for linear and nonlinear systems. In M. Luksik, C. Martin and W. Shadwick (Eds.), *Differential Geometry, the Interface between Pure and Applied Mathematics. Contemporary Mathematics*, Vol. 68, pp. 157–189. American Mathematical Society, Providence, RI.

Krener, A. J. (1989). Nonlinear controller design via approximate normal forms. Technical Report, Institute of Theoretical Dynamics and Department of Mathematics, University of California, Davis.

Krener, A. J., S. Karahan, M. Hubbard and R. Frezza (1987). Higher order linear approximations to nonlinear control systems. *Proc. 26th IEEE Conf. on Decision and Control.*, Los Angeles, CA., pp. 519–523.

Krener, A. J. and A. Isidori (1983). Linearization by output injection and nonlinear observers. *Syst. Control Lett.*, **3**, 47–52.

Krener, A. J. and W. Respondek (1985). Nonlinear observer with linearizable error dynamics. *SIAM J. Control*, **23**, 197–216.

Lee, H.-G., A. Arapostathis and S. I. Marcus (1988). On the digital control of nonlinear systems. *Proc. 27th IEEE Conf. on Decision and Control*, Austin, TX., pp. 480–481.

Lee, H.-G. and S. I. Marcus (1986). Approximate and local linearizability of nonlinear discrete-time systems. *Int. J. Control*, **44**, 1103–1124.

Monaco, S., D. Normand-Cyrot and S. Stornelli (1986). On the linearizing feedback in nonlinear sampled-data control schemes, *Proc. 25th IEEE Conf. on Decision and Control*, Athens, Greece, pp. 2056–2060.

Phelps, A. and A. J. Krener (1988). Computation of observer normal form using MACSYMA, In C. Byrnes, C.

Martin and R. Saeks, (Eds), *Nonlinear Dynamics and Control*. North Holland, Amsterdam.
Sontag, E. D. (1984). A concept of local observability, *Syst. Control Lett.*, **5**, 41–47.
Varadarajan, V. S. (1984). *Lie Groups, Lie Algebras and Their Representations*. Springer-Verlag, New York.
Xia, X. and W. Gao (1988). Nonlinear observer design by observer canonical forms. *Int. J. Control*, **47**, 1081–1100.
Xia, X. and W. Gao (1989). Nonlinear observer design by observer error linearization. *SIAM J. Control*, **27**, 199–216.

## APPENDIX A

*Proof of identity* (3.9). The following identity holds for $0 \le i, j \le n-1$

$$\langle dh \circ F^i, Ad^j_F g' \rangle (x^0) = \langle dh, Ad^{i+j}_F g' \rangle (x^0). \qquad (A.1)$$

*Proof.* Recall that $F(x^0) = x^0$ and observe that for $i = 1$, $1 \le j \le n-1$

$$\begin{aligned}
\langle dh \circ F, Ad^j_F g' \rangle (x^0) &= \frac{\partial h \circ F}{\partial x}\bigg|_{x^0} Ad^j_F g'(x^0) \\
&= \frac{\partial h}{\partial x}\bigg|_{F(x^0)} \frac{\partial F}{\partial x}\bigg|_{x^0} Ad^j_F g'(x^0) \\
&= \frac{\partial h}{\partial x}\bigg|_{x^0} \frac{\partial F}{\partial x}\bigg|_{F^{-1}(x^0)} Ad^j_F g'(F^{-1}(x^0)) \\
&= \frac{\partial h}{\partial x}\bigg|_{x^0} F_* Ad^j_F g'(F^{-1}(x^0)) \\
&= \frac{\partial h}{\partial x}\bigg|_{x^0} Ad^{j+1}_F g'(x^0) \\
&= \langle dh, Ad^{j+1}_F g' \rangle (x^0).
\end{aligned}$$

Now, suppose that the identity $(A.1)$ has been established for $i = k-1$, $1 \le j \le n-1$. Then,

$$\begin{aligned}
\langle dh \circ F^k, Ad^j_F g' \rangle (x^0) &= \frac{\partial h \circ F^k}{\partial x}\bigg|_{x^0} Ad^j_F g'(x^0) \\
&= \frac{\partial h \circ F^{k-1}}{\partial x}\bigg|_{F(x^0)} \frac{\partial F}{\partial x}\bigg|_{x^0} Ad^j_F g'(x^0) \\
&= \langle dh \circ F^{k-1}, Ad^{j+1}_F g' \rangle (x^0) \\
&= \langle dh, Ad^{k+j}_F g' \rangle (x^0). \qquad \square
\end{aligned}$$

## APPENDIX B

*Steps for proving the analyticity of $g'(x, T)$ near $T = 0$*

The lowest order term of $T$ in $\det R$ is given by

$$\begin{aligned}
&\det \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & T & T^2 & \cdots & T^{n-1} \\ \vdots & & & & \\ 1 & (n-1)T & (n-1)^2T^2 & \cdots & (n-1)^{n-1}T^{n-1} \end{vmatrix} \\
&= \det \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \\ 1 & (n-1) & \cdots & (n-1)^{n-1} \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 & \cdots \\ 0 & T & 0 & \cdots \\ 0 & 0 & \ddots & \\ 0 & & & T^{n-1} \end{vmatrix} \\
&= \det \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \\ 1 & (n-1) & \cdots & (n-1)^{n-1} \end{vmatrix} T^{(n-1)n/2}
\end{aligned}$$

and this final determinant is nonzero by a Vandermonde argument (Hoffman and Kunze, 1971).

The lowest possible order term of $T$ in $\det R(n \mid i)$ is given by

$$\begin{aligned}
&\det \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & T & \cdots & T^{i-2} & T^i & \cdots & T^{n-1} \\ \vdots & & & & & & \\ 1 & (n-1)T & \cdots & (n-1)^{i-2}T^{i-2} & (n-1)^iT^i & \cdots & (n-1)^{n-1}T^{n-1} \end{vmatrix} \\
&= (\text{constant})T^{n(n-1)/2-(i-1)}.
\end{aligned}$$

## APPENDIX C

*Proof of* (4.26)

We show that $L_{g_0}L_f^{n-1}h = 1$. Let $N_k$ be the matrix

$$N_k := \begin{bmatrix} 1 & 1/2! & \cdots & 1/k! \\ 2 & 2^2/2! & \cdots & 2^k/k! \\ \vdots & \vdots & & \vdots \\ k & k^2/2! & \cdots & k^k/k! \end{bmatrix}.$$

The Cramer's rule and a little algebra yields that

$$L_{g_0}L_f^{n-1}h = \frac{\det N_{n-2}}{\det N_{n-1}}.$$

After noting that $N_k$ can be written as the product of a Vandermonde matrix and a diagonal matrix, a simple computation gives the result.

## APPENDIX D

*Evaluating the rank of $M_i$*

To see that Matrix $M_i$ has rank $i$, for $0 \le i \le 2n-2$, take a submatrix $\bar{M}_i$ defined by

$$\bar{M}_i = \begin{bmatrix} (n-1)^i/i! & \cdots & (n-1)^2/2! & (n-1) \\ \vdots & & & \\ 2^i/i! & \cdots & 2^2/2 & 2 \\ 1^i/i! & \cdots & 1^2/2! & 1 \\ (-1)^i/i! & \cdots & (-1)^2/2! & -1 \\ (-2)^i/i! & \cdots & (-2)^2/2! & -2 \\ \vdots & & & \\ (-i+n-1)^i/i! & \cdots & (-i+n-1)^2/2! & -i+n-1 \end{bmatrix}.$$

Then,

$$\bar{M}_i = \begin{bmatrix} (n-1) & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & & -1 & & \\ & 0 & & & \ddots & \\ & & & & & (-i+n-1) \end{bmatrix} \times \begin{bmatrix} (n-1)^{i-1} & \cdots & 1 \\ \vdots & & \\ 1 & \cdots & 1 \\ (-1)^{i-1} & \cdots & 1 \\ \vdots & & \\ (-i+n-1)^{i-1} & \cdots & 1 \end{bmatrix} \times \begin{bmatrix} 1/i! & & & \\ & 1/(i-1)! & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

The second matrix has rank $i$ because it is an $i \times i$ Vandermonde matrix. Thus the submatrix $\bar{M}_i$ has rank $i$, and the proof is complete. $\square$