A resettable Kalman filter based on numerical differentiation

S. Diop, V. Fromion, J. W. Grizzle

Abstract The extended Kalman filter is known to have excellent filtering features. But its convergence is guaranteed only locally, that is, if it is initialized close enough to the true state value. Numerical differentiation based observers may be designed to secure global convergence to a neighborhood of the true state value. But when the measurements are corrupted by uncertain signals the state estimates of the latter observers are delayed. We propose, in this Communication, an observer scheme for nonlinear systems which combines these two design techniques to yield a globally exponentially converging observer.

Keywords Nonlinear observers; Controllability and observability of nonlinear systems.

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1 Introduction

The extended Kalman filter is widely used in practice mainly for its *excellent filtering* capability in presence of measurements corrupted by uncertain signals. But its convergence is *local* in the sense that it is proved, in the general case, only when the guess for the initial value of the state being estimated is close enough to the true one. Numerical differentiation based observers propose estimates from the expressions of the states in terms of the input, the output, and finitely many of their derivatives. Such expressions of the state exist by virtue of a quite weak acceptation of nonlinear observability. The estimation scheme is precisely to first estimate the data derivatives present in the state expressions, and then literally use the latter expressions to estimate the state. This approach has the advantage to potentially yield *global* state estimators with ultimate convergence. Because the data derivative estimates are usually designed without reference to the model of these data, numerical observers lack prediction capability, and thus their estimates are *delayed* whenever the data are corrupted by uncertain signals in a significant way.

We examine the combination of these two techniques into a new class of nonlinear observers. The main result describes a global exponential observer for a quite large class of nonlinear systems.

The paper is organized as follows. We first make explicit the notion of observability which is assumed in the design of numerical observers. We next recall rudiments of numerical differentiation estimation. We provide a formal proof of the local convergence of the extended Kalman filter in our deterministic setting. Finally the resettable Kalman filter is presented.

2 On the observability condition

The class of systems we consider is described by the state equations

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(t,u,x)\,, \\ y & = & h(t,u,x)\,, \end{array} \right. \tag{1}$$

where, for some $t_0 \geq 0$, and all $t \geq t_0$, $x(t) \in \mathbb{R}^n$ is the state with initial condition $x(t_0) = x_0$, $u(t) \in \mathbb{R}^m$ is the input, and $y(t) \in \mathbb{R}^p$ is the output. Some technical arguments will lead us to assume these data to be of some regularity. This will be made explicit at various moments.

The notion of observability for a system (1) that we need in the the design of numerical observers is the following.

If f and h are with *polynomial* components over some given differential field $\mathbf{k}$ then a quite adequate mathematical definition of the notion of observability we have in mind is: System (1) is *observable* if each state component is *algebraic* over the differential field extension of $\mathbf{k}$ generated by the data u, y. In other words, our observability notion requires each state component to be able to be written as a solution of a polynomial equation

$$H_i\left(x_i, u, \dot{u}, \ldots, y, \dot{y}, \ldots\right) = 0 \quad (2)$$

in x_i, and finitely many time derivatives of the data u, y, with coefficients in $\mathbf{k}$. The differential field $\mathbf{k}$ is a field of functions; it may be reduced to the field of reals, $\mathbb{R}$, if the system's equations do not explicitly depend on time. Details on this differential algebra approach of observability may be found in [8, 7]. There is proved, in particular, the equivalence of this definition to a rank condition on some sort of Lie derivatives of the output equation in (1). The polynomial assumption on f and h is of course crucial to the validity of this equivalence.

It turns out that the situation where f and h are not necessarily polynomial may be coped with by merely relaxing the polynomial requirement on H_i (this function still has to depend on only finitely many data derivatives). Historically, this nonlinear notion of observability was stated in different terms and appeared earlier in works including [6, 11, 16].

This definition raises at least three basic questions. First, what if the data assume some value $\overline{u}, \overline{y}$ such that all existing relations (2) for x_i degenerate into the trivial equation $0 = 0$? Second, what if all such relations as (2) are known to have more than one solution in x_i? Third, does the definition require more regularity of the data than the latter are in effect?

The first question is related to the singularity of the observability notion with respect to the data. The second one is about a so-called *local* character of the observability of the system we have at hand. And the third question refers to some structural coherence issue of this approach of observability. These points are thoroughly discussed in a separate communication. Here we content ourselves with the following comment on the uniqueness of the state of an observable system.

It is a matter of fact that the observability condition (2) for the system (1) may lead to more than 1 solution in x_i for a given input output data. This is a sort of local character of the observability definition. The eventual ambiguity one may be left with by the observability condition (2), of course, cannot be resolved without some supplemental information on the system which will discriminate the solutions of equation (2). We know that such extra information, if it exists, is not always practical to be included into the equations describing the system, but may be used at some point of an observer design.

It remains much desired, at least at theoretical levels, given a system, to be able to tell whether the observability conditions lead to a unique solution or not.

A system (1) is said to be *uniquely observable* if it is observable and the observability conditions (2) have *unique* solutions x_i in terms of u, y and their time derivatives.

While this definition catches the basic need in practice, it should be clear that a complete characterization of uniquely observable systems is typically out of reach. We provide a characterization of a practical subclass of systems which are uniquely observable, namely, the class of *rationally observable* systems.

A *rationally observable* system is one for which the observability conditions (2) are linear in the x_i's. The following characterization certainly generalizes to non differential algebraic systems, but we restrict ourselves to differential algebraic systems for simplicity.

Proposition 1 *A differential algebraic system (1) is rationally observable iff its defining differential field extension,* $\mathbf{k}\langle u, x, y\rangle$*, is* equal *to its external behavior differential field extension* $\mathbf{k}\langle u, y\rangle$*.*

If system (1) is rationally observable then for each state component x_i the observability condition (2) reduces to

$$x_i = \frac{h_i(u, y)}{q_i(u, y)} \tag{3}$$

where h_i and q_i are differential polynomials with coefficients in $\mathbf{k}$. The assertion is thus immediately proved.

For example, the system

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -x_2^2\,, \\ \dot{x}_2 & = & u\,, \\ y & = & x_1\,, \end{array} \right. \tag{4}$$

is rationally observable since

$$\left\{ \begin{array}{rcl} x_1 & = & y\,, \\ x_2 & = & -\dfrac{\ddot{y}}{2u}\,. \end{array} \right. \tag{5}$$

But the system

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -x_2^2\,, \\ \dot{x}_2 & = & ux_2\,, \\ y & = & x_1\,, \end{array} \right.$$

is not rationally observable since

$$\left\{ \begin{array}{rcl} x_1 & = & y\,, \\ x_2^2 & = & -\dot{y}\,, \end{array} \right.$$

and there is no means to reduce the degree of the observability condition of x_2.

3 On regularized numerical differentiation

We refer the reader to [9] for more details on numerical differentiation algorithms, their theory as well as implementation. For the sake of completeness of the present paper we provide the following as a basis for error bound analysis in regularized numerical differentiation.

We limit ourselves to the mollification approach of regularization. A *mollifier* [1, 12] is a nonnegative function φ of a single variable with integral 1 over the reals. An example is the so-called Gaussian kernel

$$\varphi(t) = \frac{1}{\sqrt{\pi}} e^{-t^2} \quad (t \in \mathbb{R}) \,.$$

Given a mollifier φ, we constructs the family $\varphi_\lambda(t) = \frac{1}{\lambda}\varphi(\frac{t}{\lambda}) \quad (t \in \mathbb{R})$, where $\lambda \in \mathbb{R}$. Mollified numerical differentiation is then defined as

$$\widehat{\dot{y}}(t) = R_\lambda y(t) = (\dot{\varphi}_\lambda * y)(t) \,,$$

where,

$$\dot{\varphi}_\lambda(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\varphi_\lambda(\tau) \,,$$

and $*$ denotes convolution of functions, and where y assumes a compact support.

The main reason for using mollification is the following set of results, which may be found in [1, 13], for instance. The filtered data $\varphi_\lambda * y$ is *infinitely differentiable.* Moreover, for the Gaussian mollifier

$$\begin{aligned} ||R_\lambda y||_{L^2} &\leq \frac{4}{\lambda\sqrt{\pi}} ||y||_{L^2} \\ ||R_\lambda K x||_{L^2} &\leq 2||x||_{L^2}; \end{aligned}$$

i.e., the family of $(R_\lambda)_{\lambda>0}$ is *uniformly bounded.* Now the regularization error with uncertain data is bounded by

$$||\dot{y} - \widehat{\dot{y}}||_{L^2} = ||x - R_\lambda \overline{y}||_{L^2} \leq 2\sqrt{2}\lambda||\dot{x}||_{L^2} + \frac{4\sigma}{\lambda\sqrt{\pi}} = 2\sqrt{2}\lambda||\ddot{y}||_{L^2} + \frac{4\sigma}{\lambda\sqrt{\pi}} \,.$$

Under the assumption $||\ddot{y}||_{L^2} \leq E$, this bound is minimized (optimal) for

$$\lambda = \lambda(\sigma) = \sqrt{\frac{2}{\sqrt{2\pi}}}\sqrt{\frac{\sigma}{E}}$$

which yields the error bound

$$||\dot{y} - \widehat{\dot{y}}||_{L^2} \leq \frac{8}{\sqrt[4]{2\pi}}\sqrt{\sigma E}$$

that is an admissible differentiation scheme for the first order differentiation operator.

This rough presentation of numerical differentiation via mollification is a quite adequate setting to understand the main problems and get an idea on how they can be circumvented. But we usually do not implement it as is. One reason is that the signal to be differentiated is available only sampled, so that we need to consider issues stemming from the approximation of the continuous time convolutions by their discrete time versions. Another reason is the memory size which is necessary to implement even the discrete time convolution.

In practice, the mollification is implemented through the fast Fourier transformation (fft), and mainly seen as a device for the regularization of the data for later use in differentiation schemes which are simpler than the above mollified differentiation one. The use of fft necessarily implies delays in the mollified data. And, as an example of a simple regularized differentiation scheme we may use the so-called Savitzky-Golay or the averaged finite difference differentiation schemes as depicted in [9]. This differentiation scheme will contribute to the overall estimation delay by an additional term, generally smaller than the delay from the mollification.

4 On numerical observers design

The basic scheme of numerical observers is to feed the observability conditions (2), and (3) with estimates of the first derivatives of u and y that are invoked in such relations. One has to bear in mind that this is only the short way of introducing the idea of numerical observers. There are, generally, a lot of *flexibility* and two main issues: the *simplicity* of the observability conditions which are selected as to be used for the state estimate, and the *differentiability of the data*. The flexibility mainly results from the fact that there are generally different presentations of the observability condition. The simplicity refers to two different aspects: one is the usual intuitive notion of simplicity which refers to numbers of operations necessary to evaluate a given quantity, and the other one stems from well-posedness of expressions. It may happen that one observability condition is ill-posed while another one is not. The differentiability of the data becomes an issue in that, if part of the data is not enough differentiable to allow standard differential field axioms

$$(a+b)^{\bullet} = \dot{a} + \dot{b}\,,$$

$$(a \cdot b)^{\bullet} = b\dot{a} + a\dot{b}\,,$$

then either there is no observability conditions in the function spaces thus considered, or we are left with more complex relations in the observability conditions. With a specific example on one's hands one has to choose the most appropriate relations as observability conditions.

Of course, as a potential alternative, one may always embed system signals in larger function spaces where the derivation operation commutes with field operations so that one recovers the simpler observability conditions.

Let us consider the example of system (4). Here, y is twice differentiable since x_2 is differentiable and $\dot{y} = -x_2^2$. If u is never small then the advantage of observability condition (5) is that it provides estimates of x_2 without further knowledge of the system. Moreover, x_2 may change sign, and the estimation scheme remains valid.

Now assume that the estimation scheme did have time to fully start functioning before u goes small so that we can no more keep on using the observability condition (5). In this situation the only remaining observability condition is

$$x_2^2 = -\dot{y}\,,$$

and, here, by continuity of x_2 (recall, x_2 is differentiable, so that it must be continuous) we know the sign of x_2, i.e, we are able to tell which one of $\sqrt{-\dot{y}}$, or $-\sqrt{-\dot{y}}$ is the correct value for x_2. Now note that, in this simple example, instead of using the latter observability condition, we might have noticed that when u is small then x_2 is approximately constant, and then we merely maintain constant our current estimate of x_2.

Finally, assume that the estimation scheme starts with u small. Then we are unable to deduce the sign of x_2 from the system's model.

Let us illustrate the potential data differentiability issue through the following simple example. Consider the system

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2 + u\,, \\ \dot{x}_2 & = & x_3\,, \\ \dot{x}_3 & = & -x_1^2 + u\,, \\ y & = & x_1\,. \end{array}\right. \tag{6}$$

If u is once differentiable then y is twice differentiable and we may easily find that

$$\left\{\begin{array}{rcl} x_1 & = & y\,, \\ x_2 & = & \dot{y} - u\,, \\ x_3 & = & \ddot{y} - \dot{u}\,. \end{array}\right.$$

But if u is not differentiable the latter observability condition of x_3 is not usable. We again easily find that we will use the following one

$$x_3 = (\dot{y} - u)^{\bullet}\,,$$

instead.

5 The local convergence of the extended Kalman filter

We consider the system

$$\left\{\begin{array}{rcl} \dot{x} & = & f(t,x,u)\,, \\ y & = & h(t,x,u)\,, \end{array}\right. \tag{1}$$

where f and h satisfy the following conditions.

Assumption 2 *f and h are $\mathcal{C}^2$. Moreover, there exist positive constants L_1, L_{2f}, L_{2h} such that for all $t \geq t_0$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$ and $i, j \in \mathbb{N}, 1 \leq i, j \leq n$*

$$\begin{cases} \|\dfrac{\partial f}{\partial x}\| < L_1 \,, \\ \|\dfrac{\partial h}{\partial x}\| < L_1 \,, \\ \|\dfrac{\partial^2 f}{\partial x_i \partial x_j}\| < L_{2f} \,, \\ \|\dfrac{\partial^2 h}{\partial x_i \partial x_j}\| < L_{2h} \,. \end{cases}$$

This assumption guarantees the existence, uniqueness and absolute continuity of the solution $x(t) = \phi(t, t_0, x_0, u)$ of system (1) for all $t \geq t_0$ and for any continuous input $u(t)$, and initial condition $x_0 \in \mathbb{R}^n$.

The extended Kalman filter (EKF) for system (1) is defined by the following specifications (7–10)

$$\dot{\widehat{x}} = f(t, \widehat{x}, u) + K(y - h(t, \widehat{x}, u)) \tag{7}$$

$$K = PH'R^{-1} \,, \tag{8}$$

$$\dot{P} = FP + PF' - PH'R^{-1}HP + Q \,, \tag{9}$$

$$\begin{cases} F = \dfrac{\partial f}{\partial x}(t, \widehat{x}, u)) \,, \\ H = \dfrac{\partial h}{\partial x}(t, \widehat{x}, u) \,, \end{cases} \tag{10}$$

with initial conditions $\widehat{x}(t_0) = \widehat{x}_0$, $P(t_0) = P_0 \geq 0$, and where $Q = Q(t)$, $R = R(t)$ and R^{-1} are assumed to be continuous, bounded, positive definite matrices.

By Assumption 2 the EKF is well-defined for all $t \geq t_0$. Classical results on differential equations [2] may be invoked to prove the existence, uniqueness and absolute continuity of the solution $\widehat{x}$ with respect to initial conditions $\widehat{x}_0$, $P_0 > 0$, for any y that is a solution of system (1) and for any finite time interval.

The pair (F, G) is said to be *uniformly controllable* if there exist positive constants α_c, β_c and σ_c satisfying, for all $t \geq t_0$,

$$\beta_c I < \int_t^{t+\sigma_c} \Phi(\tau, t+\sigma_c) G(\tau) Q(\tau) G'(\tau) \Phi'(\tau, t+\sigma_c) d\tau < \alpha_c I \,,$$

where Φ is the transition matrix of F.

If F is bounded then the pair (F, I) is known to be uniformly controllable.

The pair (F, H) is said to be *uniformly observable* if there exist positive constants α_o, β_o and σ_o satisfying, for all $t \geq t_0$,

$$\beta_o I < \int_t^{t+\sigma_o} \Phi'(\tau, t+\sigma_o) H'(\tau) R^{-1}(\tau) H(\tau) \Phi(\tau, t+\sigma_c) d\tau < \alpha_o I \,.$$

We have the following result

Lemma 3 *If the pair (F, H) is uniformly observable and the pair (F, I) is uniformly controllable then the solution of the Riccati equation* (9) *satisfies, for all $t \geq t_0 + max(\sigma_o, \sigma_c)$, the following inequality*

$$\frac{\beta_c}{1+\beta_c\alpha_o} \leq P(t) \leq \frac{1+\beta_o\alpha_c}{\beta_o}$$

The proof follows immediately from Theorem 2.1 of [4]. We actually have

$$[C_Q(t, t-\sigma)^{-1} + W_R(t, t-\sigma)]^{-1} \leq P(t) \leq W_R(t, t-\sigma)^{-1} + C_Q(t, t-\sigma)$$

for all $t \geq t_0 + \max(\sigma_o, \sigma_c)$, and

$$W_R(t_0, t_1) = \int_{t_0}^{t_1} \Phi'(\tau, t_1) H'(\tau) R^{-1}(\tau) H(\tau) \Phi(\tau, t_1) d\tau$$

and

$$C_Q(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \tau) Q(\tau) \Phi'(t_1, \tau) d\tau \;.$$

The local convergence of the EKF then is stated as follows.

Proposition 4 *If (F, H) is uniformly observable and (F, I) is uniformly controllable then the EKF is locally exponentially convergent in the sense that there are positive constants a, λ and δ such that*

$$\forall t \geq t_0, \quad \|x_0 - \widehat{x}_0\| \leq \delta \Longrightarrow \|\widehat{x}(t) - x(t)\| \leq a\|x_0 - \widehat{x}_0\| e^{-\lambda(t-t_0)} \,.$$

Moreover, an estimate of δ is given by

$$\delta = \frac{e^{-M\max(\sigma_o,\sigma_c)}\sigma_Q \beta_o^2 \beta_c}{(1+\beta_o\alpha_c)(\beta_o L_{2f} + (1+\beta_0\alpha_c) L_1 L_{2h} \sigma_{R^{-1}})(1+\beta_c\alpha_o)}$$

with $\alpha_o, \beta_o, \sigma_o \alpha_c, \beta_c$ and σ_c associated, respectively, with the uniform observability and the uniform controllability, $\sigma_Q = \|Q\|$, $\sigma_{R-1} = \|R^{-1}\|$ and $M = L_1 \sigma_{R^{-1}} (e^{2L\max(\sigma_o,\sigma_c)} \|P_0\| + \alpha_c(\frac{\sigma_c}{\sigma_o} + 1))$.

The estimation error, $\widetilde{x} = x - \widehat{x}$, assumes the following dynamics

$$\dot{\widetilde{x}} = f(t, \widehat{x}, u) + K\left(y - h(t, \widehat{x}, u)\right) - f(t, x, u)\,, \tag{11}$$

By Lemma 3, the solution P of the Riccati equation (9) has *uniform a priori* upper and lower bounds, which implies that K is also uniformly bounded with respect to time.

Expanding $f(t, x, u) + K\left(y - h(t, x, u)\right)$ around the current state estimate, $\widehat{x} = \widehat{x}(t)$ the error dynamics can be rewritten as follows

$$\left\{ \begin{array}{l} \dot{\widetilde{x}}(t) = (F - KH)\widetilde{x} + g(t, \widetilde{x}, u) \\ \widetilde{x}(t_0) = \widehat{x}_0 - x_0 \end{array} \right. \tag{12}$$

where g represents the Taylor expansion remainder which satisfies

$$\|g(t, \widehat{x} - x, u)\| \leq \frac{1}{2}\left(L_{2f} + \bar{\sigma}_K L_{2h}\right)\|\widetilde{x}\|^2$$

by Assumption 2, and the fact that $\|K\| \leq \|P\|\|H\|\|R^{-1}\| = \bar{\sigma}_K$.

We now associate with equation (11), a function V defined from $\mathbb{R}^n$ into $\mathbb{R}$ by

$$V(\widetilde{x}) = \widetilde{x}' P^{-1} \widetilde{x}\,.$$

Classical Lyapunov calculations then lead to

$$\begin{aligned} \dot{V} &= -\widetilde{x}' P^{-1} \dot{P} P^{-1} \widetilde{x} + 2\widetilde{x}' P^{-1}\left((F - KH)\widetilde{x} + g(t, \widetilde{x}, u)\right) \\ &= \widetilde{x}' P^{-1}\left(-\dot{P} + (F - KH)P + (F - KH)' P\right) P^{-1}\widetilde{x} + 2\widetilde{x}' P^{-1} g(t, \widetilde{x}, u) \\ &= \widetilde{x}' P^{-1} Q P^{-1} + 2\widetilde{x}' P^{-1} g(t, \widetilde{x}, u) \end{aligned}$$

By Lemma 3, there are σ_1, σ_2 such that $\sigma_1 \leq P^{-1} \leq \sigma_2$, thence

$$\dot{V} \leq -\sigma_1^2 \widetilde{x}' Q \widetilde{x} + \sigma_2 M \|\widetilde{x}\|^3$$

where $M = L_{2f} + \bar{\sigma}_K L_{2h}$.

This ensures that there are $\lambda > 0$ and $\delta > 0$ such that for all initial value $\|\widetilde{x}(t_0 + \max(\sigma_o, \sigma_c))\| \leq \delta$,

$$\dot{V} < -\tilde{\lambda} V$$

with $\delta = \frac{\sigma_1 \sigma_Q}{(L_{2f} + \bar{\sigma}_K L_{2h})\sigma_2}$ and for any $t \geq t_0 + \max(\sigma_o, \sigma_c)$, and for some positive $\tilde{\lambda}$.

Let $\sigma = \max(\sigma_o, \sigma_c)$. It remains to consider the interval of time between t_0 and $t_0 + \sigma$. The dynamics of equation (11) is Lipschitz and continuous:

$$\|f(t, \widehat{x}, u) - f(t, x, u) + K\left(h(t, x, u) - h(t, \widehat{x}, u)\right)\| \leq (L_1 + \sigma_K L_1)\|\widetilde{x}\|$$

since f and h are Lipschitz continuous and K is bounded. Indeed, by Lemma 5.1 of [5], we have

$$P(t_0 + \sigma) \leq \Phi(t_0 + \sigma, t_0) P_0 \Phi(t_0 + \sigma, t_0) + C_Q(t_0, t_0 + \sigma) \ .$$

Moreover $\Phi(t_0 + \sigma, t_0) \leq e^{L_1 \sigma}$ and $C_Q(t_0, t_0 + \sigma) \leq \alpha_c(\frac{\sigma_c}{\sigma_o} + 1)$. By the Bellman-Gronwall's lemma we have

$$\|\widetilde{x}(t_0 + \sigma)\| \leq \widetilde{x}(t_0) e^{M\sigma}$$

where $M = L_1 \sigma_{R^{-1}}(e^{2L\sigma}\|P_0\| + \alpha_c(\frac{\sigma_c}{\sigma_o} + 1))$. Therefore, if $\|\widetilde{x}(t_0)\| < \delta$ with $\delta = \frac{\delta}{e^{M\sigma}}$, then $x(t + \sigma) < \delta$.

The main assumption in the previous lemma is the uniform observability of the linearizations of the system along the current value of the estimate. This apparently strong assumption is actually implicitly to be satisfied by many existing nonlinear observers. We first weaken this uniform observability assumption using the following lemma which corresponds to a first step in the search of a detectability condition on the linearization (in the same way as in the linear discrete time varying case [3]).

Lemma 5 *If there exists a bounded matrix $K(t)$ such that the system:*

$$\dot{\xi} = (F - KH)\xi$$

is exponentially stable*, then the solution of the following differential Riccati equation:*

$$\dot{P} = FP + PF' - PH'R^{-1}HP + Q$$

is uniformly lower and upper bounded*.*

Since there exists an output injection gain which imposes the exponential stability of the observer, the same arguments as in [3] can be used to prove that there exists at least one output injection gain which ensures the boundedness of the cost and which is associated with the linear time varying criterion in the Kalman filter problem. Moreover, noting that the output injection gain of the Kalman filter is the optimal one, the Riccati equation is thus well-defined. By the previous discussion, its solution has both upper and lower bounds if the pair (F, Q) is controllable.

The previous lemma indicates that many classical observers implicitly satisfy the assumption of the EKF local convergence. To illustrate this point, we just consider the case of a quadratic type observers. The quadratic type observer is classical in the literature (see, e.g., [18, 17]): as a matter of fact, the main idea is to use a specific quadratic Lyapunov function to prove the exponential stability of the nonlinear observer. The same type of observer is also considered by Safonov in [15, 14], where the properties of the observer are characterized with respect to the disturbance acting on the output. More recent works [19, 10] are devoted to the search of an observability condition which *a priori* ensures the existence of quadratic observers.

Theorem 6 *If there exist a positive definite matrix P, a constant matrix K, and a positive real α such that*

$$P\left(\frac{\partial f}{\partial x}(t,x,u) - K\frac{\partial h}{\partial x}(t,x,u)\right) + \left(\frac{\partial f}{\partial x}(t,x,u) - K\frac{\partial h}{\partial x}(t,x,u)\right)' P \leq -\alpha I$$

for all $t \geq t_0$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ then the observer

$$\dot{\widehat{x}} = f(t,\widehat{x},u) + K(y - h(t,\widehat{x},u)) \tag{7}$$

initialized at t_0 is exponentially stable.

Corollary 7 *If there exists an observer satisfying the assumption of Theorem 6 then the EKF is locally exponentially stable.*

6 The resettable Kalman filter

The following theorem provides foundations for the combination of the EKF with numerical differentiation observers.

Theorem 8 *If for any $\varepsilon > 0$, there exists a finite time, t_i, such that*

$$\|x_d(t_\mathrm{i}) - x(t_\mathrm{i})\| \leq \varepsilon$$

and the system satisfies the assumption of Lemma 4 then there exists a global exponential stable observer.

Typically, x_d stands for a numerical differentiation based estimate of the state, and t_i for the delayed instant where this estimate is made. The observer will exponentially converge if it is initialized at t_i with ε at most equal to the numerical differentiation based state estimation error. We then consider the following combined observer

$$\widehat{x}(t) = \begin{cases} \widehat{x}_1(t_0), & t \in [t_0, t_\mathrm{i}) \\ \widehat{x}_2(t), & t \in [t_\mathrm{i}, \infty) \end{cases}$$

where $\widehat{x}_1(t)$ is given by equation (7) initialized at $t = t_0$ by $\widehat{x}(t_0)$ and where $K = 0$ and $\widehat{x}_1(t)$ is given by equation (7) initialized at $t = t_\mathrm{i}$ with $\widehat{x}(t_\mathrm{i}) = x_d(t_\mathrm{i})$.

For all $t \in [t_0, t_\mathrm{i})$, the estimation error, $\widetilde{x}$, satisfies the following inequalities

$$\|\widetilde{x}\| \leq \|\widehat{x}(t_0) - x(t)\| e^{L(t-t_0)} \ .$$

The error dynamics is then given by

$$\dot{\widetilde{x}} = f(t,\widehat{x},u) - f(t,x,u)$$

and one has $\|f(t,\widehat{x},u) - f(t,x,u)\| \leq L_1 \|\widetilde{x}\|$ and the upper bound by the Bellman Gronwall's lemma.

On the other hand, for all $t \geq t_\mathrm{i}$, one has, by Lemma 4,

$$\|\widehat{x}(t) - x(t)\| \leq a\|x_d(t_\mathrm{i}) - x(t_\mathrm{i})\| e^{-b(t-t_\mathrm{i})}$$

given that $\varepsilon \leq \delta_0$. Therefore, there exist $\alpha, \beta > 0$ such that

$$\|\widehat{x}(t) - x(t)\| \leq \alpha\|\widehat{x}(t_0) - x(t_0)\| e^{-\beta(t-t_0)} \ .$$

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